Today

• Binary addition
• Representing negative numbers
Binary Addition

Consider the following binary numbers:

0 0 1 0 0 1 1 0
0 0 1 0 1 0 1 1

How do we add these numbers?
Binary Addition

\[
\begin{array}{cccccc}
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 1 & 1 & 1 & 1 & 0
\end{array}
\]
Binary Addition

0 0 1 0 0 1 1 0
0 0 1 0 1 0 1 1
0 1
And we have a carry now!
Binary Addition

0 0 1 0 0 1 1 0
0 0 1 0 1 0 1 1

0 0 1

And we have a carry again!
Binary Addition

00100110
00101011
0001
and again!
Binary Addition

\[
\begin{array}{c}
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1
\end{array}
\]
Binary Addition

00100110
00101011
\[\rightarrow\]
010001

One more carry!
Binary Addition

\[
\begin{array}{c}
00100110 \\
00101011 \\
\downarrow \downarrow \\
01010001
\end{array}
\]
Binary Addition

Behaves just like addition in decimal, but:

- We carry to the next digit any time the sum of the digits is 2 (decimal) or greater
Negative Numbers

So far we have only talked about representing non-negative integers

- What can we add to our binary representation that will allow this?
Representing Negative Numbers

One possibility:

- Add an extra bit that indicates the sign of the number
- We call this the “sign-magnitude” representation
Sign Magnitude Representation

+12 0 0 0 0 1 1 0 0
Sign Magnitude Representation

+12  0 0 0 0 1 1 0 0

-12  1 0 0 0 1 1 0 0
Sign Magnitude Representation

+12: 0 0 0 0 1 1 0 0

-12: 1 0 0 0 1 1 0 0

What is the problem with this approach?
Sign Magnitude Representation

What is the problem with this approach?

- Some of the arithmetic operators that we have already developed do not do the right thing
Sign Magnitude Representation

Operator problems:
• For example, we have already designed a counter (that implements an ‘increment’ operation)

-12 1 0 0 0 1 1 0 0
Sign Magnitude Representation

Operator problems:

-12  1 0 0 0 1 1 0 0

Increment
Sign Magnitude Representation

Operator problems:

-12

1 0 0 0 1 1 0 0

Increment

1 0 0 0 1 1 0 1
Sign Magnitude Representation

Operator problems:

-12 → 1 0 0 0 1 1 0 0
  Increment

-13 → 1 0 0 0 1 1 0 1

!!!
Representing Negative Numbers

An alternative:
(a little intuition first)

0 0 0 0 0 0 0 0

Decrement
Representing Negative Numbers

An alternative:
(a little intuition first)

\[ \begin{array}{c}
0 \\
00000000 \downarrow \\
11111111
\end{array} \]

Decrement
Representing Negative Numbers

An alternative:
(a little intuition first)

Define this as

\[ \begin{align*}
0 & \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
-1 & \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1
\end{align*} \]

Decrement
Representing Negative Numbers

A few more numbers:

<p>| | | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>-2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>-3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Two's Complement Representation

In general, how do we take the additive inverse of a binary number?
Two's Complement Representation

In general, how do we take the additive inverse of a binary number?

- Invert each bit and then add '1'
Two’s Complement Representation

Invert each bit and then add ‘1’

5

0 0 0 0 0 1 0 1

Two’s complement

-5

1 1 1 1 1 0 1 1
Two’s Complement Representation

Now: let’s try adding a positive and a negative number:

\[
\begin{align*}
12 & \quad 00001100 \\
+ & \quad + \\
-5 & \quad 11111011
\end{align*}
\]
Two’s Complement Representation

Now: let’s try adding a positive and a negative number:

\[
\begin{array}{c}
12 \\
+ \\
-5 \\
\end{array}
\quad
\begin{array}{c}
0001100 \\
+ \\
1111011 \\
\end{array}
\quad
\begin{array}{c}
00000111 \\
\end{array}
\]
Two's Complement Representation

Now: let's try adding a positive and a negative number:

\[
\begin{array}{c}
12 \\
+ \\
-5 \\
\hline
7
\end{array}
\quad +
\quad
\begin{array}{c}
0001100 \\
11111011 \\
\hline
00000111
\end{array}
\]
Two’s Complement Representation

Two’s complement is used for integer representation in today’s processors
Two’s Complement Representation

Two’s complement is used for integer representation in today’s processors

One oddity: we can represent one more negative number than we can positive numbers
Implementing Subtraction

How do we implement a ‘subtraction’ operator?
(e.g., A – B)
Implementing Subtraction

How do we implement a ‘subtraction’ operator?
(e.g., A – B)

• Take the 2s complement of B
• Then add this number to A
Representing Fractions

Floating point representations are expensive:

• Require many bits
• Either require specialized hardware or long functions to compute mathematical operations
A Low-Cost Alternative: Fixed Point Representations

“w.f” fixed point:

• w bits to represent the whole number (including the sign)
• f bits to represent the fraction
A Low-Cost Alternative: Fixed Point Representations

“w.f” fixed point:
- We are representing values in units of $2^{-f}$

So: 5.3 fixed point
- 5 bits for whole
A Low-Cost Alternative: Fixed Point Representations

5.3 fixed point (fits in an int8_t)
- 5 bits for whole
- 3 bits for fraction

What can we represent with this?
A Low-Cost Alternative: Fixed Point Representations

What can we represent with 5.3 fixed point?

- 5 bits for whole: 15 ... -16
- 3 bits for fraction: units of 1/8th
<table>
<thead>
<tr>
<th>Fixed Point</th>
<th>Value</th>
<th># of eighths</th>
</tr>
</thead>
<tbody>
<tr>
<td>00000 000</td>
<td>0.0</td>
<td>0 eighths</td>
</tr>
<tr>
<td>00000 001</td>
<td></td>
<td></td>
</tr>
<tr>
<td>00000 100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>00001 000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>00101 010</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Fixed-Point Example

<table>
<thead>
<tr>
<th>Fixed Point</th>
<th>Value</th>
<th># of eighths</th>
</tr>
</thead>
<tbody>
<tr>
<td>000000 000</td>
<td>0.0</td>
<td>0 eighths</td>
</tr>
<tr>
<td>000000 001</td>
<td>0.125</td>
<td>1 eighth</td>
</tr>
<tr>
<td>000000 100</td>
<td>0.5</td>
<td>4 eighths</td>
</tr>
<tr>
<td>000001 000</td>
<td>1.0</td>
<td>8 eighths</td>
</tr>
<tr>
<td>001010 010</td>
<td>5.25</td>
<td>42 eighths</td>
</tr>
</tbody>
</table>
Adding Fixed-Point Numbers

```
int8_t a = 5;    // 5/8
int8_t b = 10;   // 10/8
int8_t c = a + b ???
```

\[ 5 \left( \frac{1}{8}s \right) + 10 \left( \frac{1}{8}s \right) = 15 \text{ what?} \]
Adding Fixed-Point Numbers

```c
int8_t a = 5;    // 5/8
int8_t b = 10;   // 10/8
int8_t c = a + b; // 15/8
```

\[
5 \left( \frac{1}{8}s \right) + 10 \left( \frac{1}{8}s \right) = 15 \left( \frac{1}{8}s \right)
\]

So: addition does the right thing
Multiplying Fixed-Point Numbers

```c
int8_t a = 5; // 5/8
int8_t b = 10; // 10/8
int8_t c = a * b ???
```

\[
5 \left( \frac{1}{8} s \right) \times 10 \left( \frac{1}{8} s \right) = 50 \text{ what?}
\]
Multiplying Fixed-Point Numbers

\[
\begin{align*}
\text{int8_t } a &= 5; \quad \text{// } 5/8 \\
\text{int8_t } b &= 10; \quad \text{// } 10/8 \\
\text{int8_t } c &= a \times b \quad \text{??？}
\end{align*}
\]

\[
5 \left( \frac{1}{8} s \right) \times 10 \left( \frac{1}{8} s \right) = 50 \left( \frac{1}{64} s \right)
\]

But: we need to keep things in 5.3 format
Multiplying Fixed-Point Numbers

```c
int8_t a = 5;   // 5/8
int8_t b = 10;  // 10/8
int8_t c = (a * b) >> 3;  // 6/8
```

\[
5 \left( \frac{1}{8}s \right) \times 10 \left( \frac{1}{8}s \right) = 50 \left( \frac{1}{64}s \right) \approx 6 \left( \frac{1}{8}s \right)
\]
Dividing Fixed-Point Numbers

```c
int8_t a = 20;    // 20/8
int8_t b = 7;     // 7/8
int8_t c = a / b ???
```

\[
20 \left( \frac{1}{8}s \right) \div 7 \left( \frac{1}{8}s \right) = 2 \quad \text{What?}
\]
Dividing Fixed-Point Numbers

```c
int8_t a = 20;    // 20/8
int8_t b = 7;     // 7/8
int8_t c = a / b  // ???
```

\[
20 \left( \frac{1}{8} s \right) \div 7 \left( \frac{1}{8} s \right) = 2 \ (1s)
\]

But: we want to stay within the 5.3 format. And – note that we have lost information in the rounding!
Dividing Fixed-Point Numbers

```c
int8_t a = 20;    // 20/8
int8_t b = 7;     // 7/8
int8_t c = (a << 3) / b;  // 160/7

20 \left(\frac{1}{8}s\right) \div 7 \left(\frac{1}{8}s\right) = 22 \left(\frac{1}{8}s\right)
```
Notes About the Book

The example code that the book gives tries to address some additional questions (but fails to be clear):

• In conversions from floating point to fixed-point, it catches errors when a floating point value is too small or too large to fit in the fixed point representation

• `assert(0)` just means that an error should be generated
Notes About the Book

- In the book, a "short" is 16 bits and a "long" is 32 bits.
- For many of the fixed-point examples, the fixed-point values fit in 16 bits.
- After we perform a mathematical operation, it is possible that the result will not fit within the 16 bits.
- So: all numbers are converted to 32 bits before the operation & the results are checked before converting back to 16 bits.