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Network Science Meets Circuit Theory: Resistance Distance, Kirchhoff Index, and Foster's Theorems With Generalizations and Unification

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***Abstract*— The emerging area of network science and engineering is concerned with the study of structural characteristics of networks, their impact on the dynamical behavior of systems as revealed through their topological properties, random evolution of networks, information spreading along a network, and so on. This area spans a wide range of applications in different disciplines. A topic of great interest in this area is the notion of network criticality. Most measures of network criticality are defined by the paths that flow through the nodes or edges. Since computing all the paths is computationally intractable, only the shortest paths are usually used for computing criticality metrics. Thus, measures that implicitly capture the impact of all the paths will be useful. The recently introduced concepts of the resistance distance and the Kirchhoff Index are two such measures. In this paper, we study these metrics and present several results that extend, generalize, and unify earlier works reported in the literature. In developing these results, the role of circuit theoretic concepts is emphasized. We also relate our works to Foster's theorems and present a generalization that captures and retains the circuit theoretic elegance of Foster's original theorems.**

***Index Terms*— Foster's theorems, Kirchhoff Index, network centrality measures, resistance distance.**

I. Introduction

THE emerging area of network science and engineering is concerned with the study of structural characteristics of networks, their impact on the dynamical behavior of systems as revealed by their topological properties, random evolution of networks, and information spreading along a network (called epidemic process) etc. This area spans a wide range of applications in different disciplines, including electrical, computer and communication networks, road networks, biological networks, information networks (such as world wide web), financial networks etc. For examples see [1]–[5]. It is therefore not surprising that researchers in different disciplines have contributed to the developments in network science and engineering. In particular, graph theory which includes structural graph theory, random graphs, spectral graph theory, and algebraic graph theory, plays a central role in network science and engineering.

Our focus in this paper is on certain issues that arise in the study of structural characteristics of network such as edge and node betweenness, similarity, node ranking, communities in social networks etc. In this area, several criticality metrics that capture the importance and impact of nodes and links have been introduced [1]. In defining these measures, paths between nodes are often used. Since the number of paths between any two nodes is exponentially large, only shortest paths are considered. But the shortest paths do not adequately capture the impact of all the paths. So, measures that implicitly use all the paths will be useful. Resistance distance between a pair of nodes and Kirchhoff Index which is the aggregate of all resistance distances across all pairs of nodes are two such measures.

The concept of resistance distance, though well known in circuit theory literature since the early years of electrical engineering, was first introduced in [6] in the context of chemical graph theory. See also [7] and [8] for some related works and references. The concept of Kirchhoff Index was introduced in [6] as a descriptor of molecular structures. See [8] and [9] for a discussion of this concept and several other graph invariants of interest in chemical graph theory. See [10]–[12] for an application of resistance distance in network meta data analysis.

Given a weighted undirected graph G, treating G as a resistance network with edge conductances equal to corresponding weights, [13] discusses relationship between random walks and flow of currents in G. It has been shown that the average number of edges traversed by a random walk, that starts at a node u and ends at a node v, is proportional to the resistance distance between u and v. See [14] for a proof of this result. See [15] for an application of the concept of Kirchhoff Index in congestion control in a communication network. An optimization problem in the context of shortest path routing is studied in [16].

In [17] Foster presented a theorem which gives an invariant involving resistance distances. This invariant does not depend on the edge weights (conductances). A generalization

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of this theorem called Foster's second theorem, was given by Foster in [18]. An application of Foster's theorem in the analysis of on-line algorithms is given in [19]. A new proof of Foster's second theorem is given in [20] using a probabilistic approach. Extensions of Foster's theorems were given in [21]–[24]. Foster's theorem and its extension was also encountered in [10] and [12] in the context of network meta data analysis and combinatorics of optimal designs. The extension given in [12] is similar to the one given in [23] and [24], but deals with special situations. However, these extensions do not capture the circuit theoretic elegance of Foster's theorems in the sense that the relevant summation in these generalizations is a function of the edge weights in the network unlike in Foster's theorems where it is an invariant independent of the edge weights.

In this paper, our focus is on resistance distance and Kirchhoff Index. We provide results that generalize and unify current results in the literature.

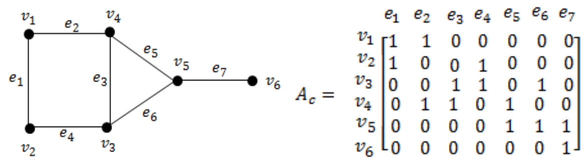
The rest of the paper is organized as follows. To make the paper self-contained, we present in sections II - IV the basic graph and circuit theoretic concepts and results (with appropriate references) needed for the development in the remainder of the paper. In particular, in section IV we define the concept of resistance distance and present a topological formula to compute this measure. The rest of the paper contains our contributions: new concepts and results. In section V the concept of Kirchhoff Index is defined and a formula to compute Kirchhoff Index starting from the inverse of a reduced Laplacian matrix of a graph is given. In section VI the concept of cutset Laplacian matrix is defined and two approaches to compute Kirchhoff Index starting from the inverse of a cutset Laplacian matrix are given. In this context we also introduce the concept of Kirchhoff polynomial of a graph. In section VII we introduce the concept of weighted Kirchhoff Index and using the results in section VI we present a formula to compute the weighted Kirchhoff Index. Section VIII presents a new generalization of Foster's theorems capturing the circuit theoretic elegance of Foster's original results. Section IX summarizes our contributions advancing the state of the art in this area and point out the generalizations and unification achieved. We also provide some topics for further research.

II. Basic Concepts and Results

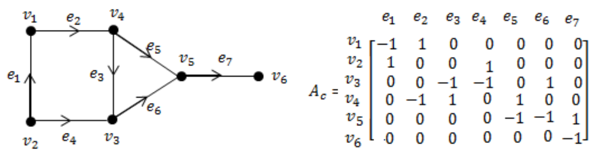
In this section, we introduce the incidence, adjacency, circuit/cutset and Laplacian matrices of a graph and highlight certain properties of these matrices that help to reveal the structure of a graph. The Laplacian matrix is derived from the incidence and the adjacency matrices and gives a representation of a graph as viewed from its external ports.[1] In electrical circuit theory literature, the Laplacian matrix is known as the *indefinite conductance matrix* and its reduced version known as *the node-to-datum conductance matrix.*

For proofs of all the results stated in this section and for graph theoretic concepts relevant to circuit theory that are not covered here [25] may be consulted. Throughout the paper, the terms links and edges, and the terms nodes and vertices will be used interchangeably.

[1]In circuit theory a port refers to a pair of vertices (called terminals) in a given circuit N. An electrical circuit is connected to the rest of a larger system through the port terminals. Ports are additional edges added to N. By connecting a current or voltage source at the terminals of ports measurements at the terminals are made. A matrix description based on these measurements characterizes the network as viewed from the ports.



(*a*)



(*b*)

Fig. 1. Incidence matrix. (a) An undirected graph G and it's all-vertex incidence matrix. (b) A directed graph G and it's all-vertex incidence matrix.

A. Incidence Matrix

Consider a graph G with n vertices and m edges and having no self-loops. The *all-vertex incidence matrix* $A_c = [a_{ij}]$ of G has n rows, one for each vertex, a, one or each edge. The element a_{ij} of A_c is defined as follows:

G is *undirected:*

$$a_{ij} = \begin{cases} 1, & \text{If the } j \text{ th edge is incident on the} \\ & i\text{th vertex;} \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

G is *directed*:

$$a_{ij} = \begin{cases} 1, & \text{if the } j \text{ th edge is incident on the } i \text{ th} \\ & \text{vertex and oriented away from it;} \\ -1, & \text{if the } j \text{ th edge is incident on the } i \text{ th} \\ & \text{vertex and oriented toward it;} \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

A row of A_c will be referred to as an *incidence vector* of G. Two graphs and their all-vertex incidence matrices are shown in Figures 1(a) and 1(b).

It should be clear that we can obtain any row of A_c from the remaining $n-1$ rows. Thus, the rows of A_c are lineraly dependent.

Any $(n-1) \times m$ submatrix A of A_c will be referred to as an *incidence matrix* of G. The vertex which corresponds to the row of A_c which is not in A will be called the *reference vertex* or *datum vertex* of A.

Theorem 1 [25]: The determinant of any incidence matrix of a tree is equal to ± 1. ∎

Since a connected graph has at least one spanning tree, we have the following.

Theorem 2 [25]: The rank of the all-vertex incidence matrix of an n-vertex connected graph G is equal to $n-1$, the rank $\rho(G)$ of G. ∎

B. Adjacency Matrix

Let $G = (V, E)$ be a directed graph with no parallel edges or self-loops. Let $V = \{v_1, v_2, \cdots, v_n\}$. The *adjacency matrix* $M = [m_{ij}]$ of G is an $n \times n$ matrix with m_{ij} define as follows:

$$m_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E. \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

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In the case of an undirected graph, $m_{ij} = 1$ only if there is an edge connecting v_i and v_j. The undirected graph of Figure 1(a) has the following adjacency matrix:

$$M = \begin{array}{c c} & \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

C. Laplacian Matrix

Let $G = (V, E)$ be a weighted undirected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n$ and edge set $E(G)$. Let w_{ij} denote the weight of edge (i, j). The adjacency matrix $M(G)$ is as defined in (3). The degree matrix $D(G)$ is defined as

$$D_{i,j} = \begin{cases} \text{sum of the weights of the} & if\ i = j \\ \text{edges incident on } i, & \\ 0, & otherwise. \end{cases} \tag{4}$$

Note that if each $w_{ij} = 1$, then $D_{i,i}$ is equal to the degree of i.

The Laplacian matrix of a connected weighted undirected graph G is a square matrix of order n, defined by

$$L(G) = D(G) - M(G). \tag{5}$$

Note that, the (i, j)- entry of the Laplacian matrix L can be written as:

$$L_{i,j} = \begin{cases} -w_{ij}, & if\ i \neq j\ and\ v_i\ and\ v_j \\ & are\ adjacent. \\ 0, & if\ i \neq j\ and\ v_i\ and\ v_j \\ & are\ not\ adjacent. \\ \text{Sum of the weights of} & if i = j. \\ \text{the edges incident on} & i, \end{cases} \tag{6}$$

So, L can also be written as

$$L = A_c W A_c^t, \tag{7}$$

where W is the diagonal matrix with the diagonal entries representing the weights on the edges. Note that the matrix A_c in (7) is obtained after assigning arbitrary orientations to the given undirected graph G.

Let $L(\bar{i})$ be the submatrix of the Laplacian matrix which is obtained by removing the ith row and the ith column from L. This matrix $L(\bar{i})$ is called a *reduced Laplacian matrix* of G and is given by

$$L(\bar{i}) = AWA^t \tag{8}$$

where A is the reduced incidence matrix with node i as reference.

The Laplacian matrix and a reduced Laplacian matrix of the weighted graph G (Figure 1) are calculated as follows.

$$W = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$$L = \begin{bmatrix} 5 & -2 & 0 & -3 & 0 & 0 \\ -2 & 5 & -3 & 0 & 0 & 0 \\ 0 & -3 & 6 & -2 & -1 & 0 \\ -3 & 0 & -2 & 6 & -1 & 0 \\ 0 & 0 & -1 & -1 & 5 & -3 \\ 0 & 0 & 0 & 0 & -3 & 3 \end{bmatrix}$$

$$L(\bar{6}) = \begin{bmatrix} 5 & -2 & 0 & -3 & 0 \\ -2 & 5 & -3 & 0 & 0 \\ 0 & -3 & 6 & -2 & -1 \\ -3 & 0 & -2 & 6 & -1 \\ 0 & 0 & -1 & -1 & 5 \end{bmatrix}.$$

Note that the Laplacian matrix is defined for an undirected graph. It can be obtained using equation (5) or (7).

D. Co-Factors of the Reduced Laplacian Matrix

Given a spanning tree of a connected graph G, the product of all the weights of edges in the spanning tree is called *the tree weight product*. We denote by $\tau(G)$ the sum of the weight products of all spanning trees of G.

Theorem 3 [25]: Let G be a connected and weighted undirected graph and A be an incidence matrix of the directed graph that is obtained by assigning arbitrary orientations to the edges of G.Then

$$\tau(G) = det(AWA^t) = det L(\bar{i}), \quad \text{for any vertex } i.$$

■

From the above theorem, one can see that every (i, i) cofactor of the Laplacian matrix is equal to $\tau(G)$.[2] In fact, we have the following result originally due to Kirchhoff [26].

Theorem 4 [25]: All the cofactors of the Laplacian matrix of a connected undirected graph G has the same value equal to $\tau(G)$. ■

Ak-tree is an acyclic graph consisting of k components. If a k-tree is a spanning subgraph of a graph G, then it is called a *spanning k-tree* of G. The spanning 2-tree $T_{ijk\ldots,rst\ldots}$ denotes a 2-tree, in which the vertices $v_i, v_j, v_k, \ldots$ are in one component and the vertices $v_r, v_s, v_t, \ldots$ are in the other component of the 2-tree and $T_{ijk,\ldots,rst}$ denotes the sum of the weight products of all such 2-trees.

Theorem 5 [22]: For a connected graph G, let Δ_{ij} denote the (i, j) cofactor of $L(\bar{k})$ for any k. Then

$$\Delta_{ii} = \tau_{i,k} \text{ and}$$
$$\Delta_{ij} = \tau_{ij,k}.$$

■

E. Pseudo-Inverse of Laplacian Matrix

It can be seen that the sum of the elements in each row and the sum of the elements in each column of a Laplacian matrix is zero, that is,

$$\sum_{i=1}^{n} L_{ij} = \sum_{j=1}^{n} L_{ij} = 0. \tag{9}$$

So, the determinant of $L(G)$ is zero and $L(G)$ has no inverse. This has led to the definition of the Moore-Penrose pseudoinverse of $L(G)$. We wish to note that whereas the pseudo inverse has been extensively studied by the mathematics community, the inverse of the reduced Laplacian matrix

[2]The (i, j) cofactor of an $n \times n$ matrix, denoted as Δ_{ij}, is equal to the product of-$(1)^{i+j}$ and determinant of $A_{ij,}$, where A_{ij} is the matrix that results after removing row i and column j from A.

(same as the node-to-datum conductance matrix) has been extensively studied and used in the electrical circuit theory literature. See [25].

The Moore-Penrose pseudoinverse of the Laplacian matrix $L(G)$ denoted by $L^{+}(G)$ can be computed as follows [8].

$$L^{+}(G) = \left(L(G) + \frac{J}{n} \right)^{-1} - \frac{J}{n} \tag{10}$$

where $J \in R^{nxn}$ is a matrix of all 1's and n is the number of vertices of graph G.

F. Cuts, Cutsets and Fundamental Matrices

A *cutset* S of a connected graph G is a minimal set of edges of G such that its removal from G disconnects G, that is, the graph $G– S$ is disconnected. Note that the minimality requirement in the definition of a cutset requires that $G - S$ has exactly two components.

Consider a connected graph G with vertex set V. Let V_1 and V_2 be two mutually disjoint subsets of V such that $V = V_1 \cup V_2$; that is, V_1 and V_2 have no common vertices and together contain all the vertices of V. Then the set S of all those edges of graph G having one end vertex in V_1 and the other in $V_2$is called a *cut* of G. This is usually denoted by $\langle V_1, V_2 \rangle$.

Suppose that for a cutset S of G, V_1 and V_2 are, respectively, the vertex sets of the two components G_1 and G_2 of $G – S$. Then S is the cut $\langle V_1, V_2 \rangle$.

Theorem 6 [25]: A cut in a connected graph G is a cutset or union of edge-disjoint cutsets of G. ■

In general, a graph has a large number of cuts as well as a large number of circuits. But there exists a set of $n - 1$ cutsets called *fundamental cutsets* that can be used to generate all the cutsets in the graph. Similarly, there exists a set of $m - n + 1$ circuits, called *fundamental circuits*, that can be used to generate all the circuits in the graph.

G. Fundamental Cutsets and Fundamental Cutset Matrix

Consider a spanning tree T of a connected graph G. Let b be a branch of T (*Note:* The edges of a spanning tree T are called *the branches of* T and all other edges are called *chords* of T). Now, the removal of the branch b disconnects T into exactly two components T_1 and T_2. Note that T_1 and T_2 are trees of G. Let V_1 and V_2, respectively, denote the vertex sets of T_1 and T_2. V_1 and V_2 together contain all vertices of G.

Let G_1 and G_2 be, respectively, the induced subgraphs of G on the vertex sets V_1 and V_2. It can be seen that T_1 and $T_2$are, respectively, the spanning trees of G_1 and G_2. Hence, G_1 and G_2 are connected. This, in turn, proves that the cut $\langle V_1, V_2 \rangle$ is a cutset of G. This cutset is known as the *fundamental cutset* of G with respect to the branch b of the spanning tree T of G. The set of all the $n - 1$ fundamental cutsets with respect to the $n - 1$ branches of a spanning tree T of a connected graph G is known as the set of *fundamental cutsets* of G with respect to the spanning tree T.

A graph G and the fundamental cutsets with respect to branches e_4 and e_3 are shown in Figure 2.

To define the fundamental cutset matrix of a directed graph we first assign an orientation to each cutset of the graph.

Given a spanning tree T of an n-vertex connected graph G, let $b_1, b_2, \ldots, b_{n-1}$ denote the branches of T. The fundamental matrix Q_f is defined as follows:

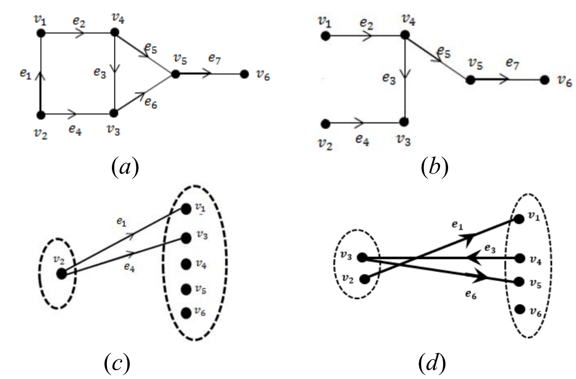


Fig. 2. Fundamental cutsets of a graph. (a) Graph G. (b) Spanning tree T of G. (c) Fundamental cutset with respect to branch e_4. (d) Fundamental cutset with respect to branch e_3.

1) Q_f has $n-1$ rows, one row for each fundamental cutset, and m columns, one column for each edge.
2) The ith row corresponds to the fundamental cutset defined by b_i.
3) $q_{ij} = \begin{cases} 1, & \text{if the } j \text{ th edge is in the } i \text{ th cut and its orientation agrees with the cut orientation;} \\ -1, & \text{if the } j \text{ th edge is in the } i \text{ th cut and its orientation does not agrees with the cut orientation;} \\ 0, & \text{otherwise.} \end{cases}$

If in addition, we assume that the orientation of a fundamental cutset is so chosen as to agree with that of the defining branch, then the matrix Q_f can be displayed in a convenient form as follows:

$$Q_f = \left[U | \, Q_{fc} \right] \tag{11}$$

where U is the unit matrix of order $n-1$ and its columns correspond to the branches of T; the columns of Q_{fc} correspond to the chords of T. Clearly the rank of Q_f is $n-1$. It is known that all the cuts can be generated using the rows of Q_f.

For example, the fundamental cutset matrix Q_f of the connected graph of Figure 2(a) with respect to the spanning tree $T = \{e_2, e_3, e_4, e_5, e_7\}$ in Figure 2 (b) is

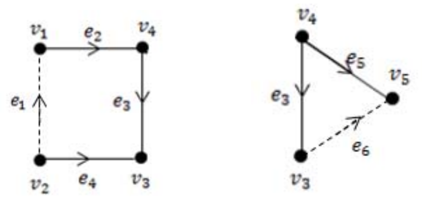
$$Q_f = \begin{array}{c} \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_2 & e_3 & e_4 & e_5 & e_6 & e_1 & e_6 \end{array} \\ \left[\begin{array}{ccccc|cc} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right] \end{array}$$

H. Fundamental Circuits and Fundamental Circuit Matrix

Consider a spanning tree T of a connected graph G. Let the branches of T be denoted by $b_1, b_2, \cdots, b_{n-1}$, and let the chords of T be denoted by $c_1, c_2, \cdots, c_{m-n+1}$, where n is the number of vertices in G and m is the number of edges in G.

While T is acyclic, the graph $T \cup c_i$ contains exactly one circuit C_i. This circuit consists of the chord c_i and those branches of T which lie in the unique path in T between the end vertices of c_i. The circuit C_i is called the *fundamental circuit* of G with respect to the chord c_i of the spanning tree T.

The set of all the $m - n + 1$ fundamental circuits $C_1, C_2, \cdots, C_{m-n+1}$ of G with respect to the chords of the



(*a*) (*b*)

Fig. 3. Two fundamental circuits of G (given in Fig. 2(a)) with respect to the spanning tree T (given in Fig. 2(b)). (*a*) Circuit C_1. (*b*) Circuit C_6.

spanning tree T of G is known as the *set of fundamental circuits* of G with respect to T.

For the graph G and its spanning tree T in Figure 2, two fundamental circuits are shown in Figure 3.

A circuit can be traversed in one of two directions, clockwise or anticlockwise. The direction we choose for traversing a circuit defines its orientation.

Consider an edge e which has v_i and v_j as its end vertices. Suppose that this edge is oriented from v_i to v_j and that it is present in circuit C. Then we say that the orientation of e agrees with the orientation of the circuit if v_i appears before v_j when we traverse C in the direction specified by its orientation.

To define the fundamental circuit matrix, consider any spanning tree T of a connected graph G having n vertices and m edges. Let $c_1, c_2, \cdots, c_{m-n+1}$ be the chords of T. We know that these $m-n+1$ chords define a set of $m-n+1$ fundamental circuits. The fundamental circuit matrix B_f with respect to the spanning tree T is defined as follows.

(i) B_f has $m-n+1$ rows, one row for each fundamental circuit, and m columns, one column for each edge.

(ii) The ith row corresponds to the fundamental circuit defined by c_i.

(iii) $$b_{ij} = \begin{cases} 1, & \text{if the } j\text{th edge is in the } i\text{th circuit and} \\ & \text{its orientation agrees with the circuit} \\ & \text{orientation;} \\ -1, & \text{if the } j \text{ th edge is in the } i \text{ th circuit and} \\ & \text{its orientation does not agrees with the} \\ & \text{circuit orientation;} \\ 0, & \text{otherwise.} \end{cases}$$

If in addition, we assume that the orientation of a fundamental circuit is so chosen as to agree with that of the defining chord, then the matrix B_f can be displayed in a convenient form as follows:

$$B_f = \left[B_{ft} \middle| U \right] \tag{12}$$

where U is the unit matrix of order $m-n+1$ and its columns correspond to the chords of T; B_{ft} is the submatrix with its columns corresponding to the branches of T.

For example, the fundamental circuit matrix of the graph of Figure 2 (a) with respect to the spanning tree $T = \{e_2, e_3, e_4, e_5, e_7\}$ is as given below:

$$B_f = \begin{array}{c} \\ e_1 \\ e_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_3 & e_4 & e_5 & e_7 & e_1 & e_6 \end{array} \\ \left[\begin{array}{ccccc|cc} 1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

It is obvious from (12) that the rank of B_f is equal to $m-n+1$, the nullity $\mu(G)$ of G. It can be shown that all the circuits in a graph can be generated using the fundamental circuits.

It is known [25] that circuit and cutset vectors are orthogonal. That is,

$$Q_f B_f^t = 0. \tag{13}$$

Using this relation, we get

$$B_{ft} = -Q_{fc}^t. \tag{14}$$

III. Kirchhoff's Laws and Fundamental Circuit and Cutset Matrices

Consider an electrical resistance network G that may have independent current and voltage sources. Let T be a spanning tree of G. Then the fundamental cutset matrix Q_f of G has the form

$$\begin{array}{c} \leftarrow \text{Branches} \rightarrow \leftarrow \text{Chords} \rightarrow \\ Q_f = \left[\quad U \quad | \quad Q_{fc} \right] \end{array}$$

and Kirchhoff's current law equations can be written as

$$Q_f I_e = 0 \tag{15}$$

that is,

$$\begin{bmatrix} U & Q_{fc} \end{bmatrix} \begin{bmatrix} I_b \\ I_c \end{bmatrix} = 0. \tag{16}$$

where I_b is the vector of branch currents and I_c is the vector of chord currents. So

$$I_b = -Q_{fc} I_c. \tag{17}$$

Also,

$$B_f = \begin{bmatrix} B_{ft} & U \end{bmatrix} = \begin{bmatrix} -Q_{fc}^t & U \end{bmatrix} \tag{18}$$

and Kirchhoff's voltage law equations can be written as

$$B_f V_e = 0, \tag{19}$$

that is,

$$\begin{bmatrix} -Q_{fc}^t & U \end{bmatrix} \begin{bmatrix} V_b \\ V_c \end{bmatrix} = 0. \tag{20}$$

where V_b is the vector of branch voltages and V_c is the vector of chord voltages. So

$$V_c = Q_{fc}^t V_b. \tag{21}$$

Note that in (15) we can use the reduced incidence matrix A defined in section I(A) in place of Q_f to write the Kirchhoff current equations. That is, we can also write

$$A I_e = 0. \tag{22}$$

IV. Resistance Distance and a Topological Formula

Consider an n -port resistance network N with $n+1$ nodes. Each port is defined by a pair of nodes. We assume that the n ports form a star tree structure. The network is available for connections through the ports to the other parts of a system. Let the $n+1$ nodes be denoted by 0, 1, 2, $\ldots$, n, and let the nodes i and 0 be, respectively, the positive and negative reference terminals of the port i.

Let us now excite the network by connecting a current source of value I_j across each port j as shown in Figure 4. Let $V_1, V_1, V_2, \ldots, V_n$ denote the voltages of the nodes 1,

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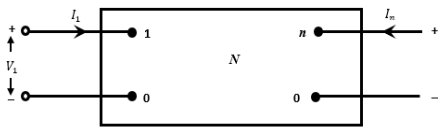

Fig. 4. An n-port network.

$2, \ldots, n$ with respect to node 0. This means $V_0 = 0$ and V_i is the voltage between the nodes i and 0 (that is $V_i = V_i - V_0$) for $i \neq 0$. Also, the A matrix does not contain the row corresponding to the node 0.

Note that in the graph representation of each port j, the corresponding edge will be oriented from the positive terminal to the negative terminal. So, the current flowing through this edge in the direction of the orientation is $-I_j$ where the voltage from positive terminal to the negative terminal of the port is V_j.

Then we have

$$AI_e - I = 0,$$

that is,

$$AI_e = I \tag{23}$$

where

$$I = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix}.$$

Let the network elements be labeled as $e_1, e_2, \ldots, e_m$ with r_i denoting the resistance value of element e_i. Then the conductance of e_i is given by $w_i = \frac{1}{r_i}$. Let W be the diagonal matrix with its (i, i) entry equal to w_i. Then we can write

$$I_e = WV_e. \tag{24}$$

Suppose the end vertices of e_i are k and l. Then the voltage across this element (voltage drop from node k to node l) is given by $V_k - V_l$, assuming that the element is oriented from vertex k to vertex l. So, we can write

$$V_e = A^t V \tag{25}$$

where V is the vector of voltages $V_1, V_2, \ldots, V_n$. Combining (23), (24) and (25) we get the node equations

$$AWA^t V = I \tag{26}$$

where

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_n \end{bmatrix}.$$

Let

$$Y = AWA^t$$

so that

$$YV = I. \tag{27}$$

Note that the matrix Y is the same as the reduced Laplacian matrix $L(\bar{0})$ defined in section I.C. We denote $Z = Y^{-1}$.

In circuit theory literature, the matrix Y is called the node-conductance matrix of the network with vertex 0 as the reference. Viewed as an n-port network, Y and $Z$are also known as the short-circuit conductance and open-circuit resistance matrices of the n-port network. Solving (27) for V_i and assigning $I_j = 0$, for all $j \neq i$, we get

$$V_i = \frac{\Delta_{ii}}{\Delta} I_i,$$

where

$$\Delta = \det Y$$

and

$$\Delta_{ii} = \text{(i, i) cofactor of } Y.$$

So, the driving-point resistance across vertices i and 0 is given by

$$z = \frac{\Delta_{ii}}{\Delta}, \tag{28}$$

and the driving-point conductance across i and 0 is given by

$$y = \frac{1}{z} = \frac{\Delta}{\Delta_{ii}}. \tag{29}$$

The resistance z defined in (28) is called the ***resistance distance*** between nodes i and 0. In general. the resistance distance between nodes i and j is denoted by r_{ij}.

The (i, j) element z_{ij} of the matrix Z is called the *transfer resistance* between ports i and j and is given by

$$\begin{aligned} z_{ij} &= \frac{V_i}{I_j}, \quad \text{when } I_k = 0 \text{ for all } k \neq j. \\ &= \frac{\Delta_{ij}}{\Delta} \end{aligned}$$

From Theorem 4 and 5, we have

$$\begin{aligned} \Delta &= \tau(G) \\ \Delta_{ii} &= \tau_{i,0} \end{aligned}$$

and

$$\Delta_{ij} = \tau_{ij,0}$$

Thus, we get the following.

Theorem 7:

$$\begin{aligned} r_{ij} &= \frac{\tau_{i,j}}{\tau(G)} \\ z_{ij} &= \frac{\tau_{ij,0}}{\tau(G)} \end{aligned} \tag{30}$$

■

Note that in the expression for r_{ij}, the reference node does not appear. But in the expression of z_{ij}, the reference node 0 appears because the matrix Z is defined with respect to a reference.

V. Kirchhoff Index of a Graph

Consider a weighted connected undirected graph G. This graph can be viewed as a resistance network N with conductances of resistance elements equal to the weights of the corresponding edges of G. The ***Kirchhoff Index*** $KI(G)$ of G is defined as

$$KI(G) = \sum_{i>j} r_{ij}.$$

In this section, we develop a formula for $KI(G)$.

In circuit theory literature, the graph representation of a network N is also referred to as N. So, in the rest of the

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paper the terms graph G and the corresponding network N will be used interchangeably.

Let $Y = [y_{ij}]$ denote the node conductance matrix of$G$ with node n as the reference or datum node. Note that Y is a square matrix of order $n-1$ and it is the matrix obtained by removing the n^{th} row and the n^{th} column from the Laplacian matrix of G. Let $Z = Y^{-1}$.

Theorem 8: $KI\,(G) = nTr\,(Z) - \sum_{k,l} z_{kl}$.

where Z is the inverse of the Laplacian matrix obtained by deleting any ith row and ith column, *TR(Z)* is the matrix Z and $\sum_{k,l} z_{kl}$ is the sum of all the elements of matrix Z (note that $Z = Y^{-1}$).

Proof: As we have seen before in Theorem 7,

$$r_{ij} = \frac{\tau_{i,j}}{\tau(G)}$$

However,

$$\begin{aligned}\tau_{ij} &= \tau_{i,nj} + \tau_{in,j}\\ &= \{\tau_{i,n} - \tau_{ij,n}\} + \{\tau_{j,n} - \tau_{ij,n}\}\\ &= \tau_{i,n} + \tau_{j,n} - 2\tau_{ij,n}\end{aligned}$$

Dividing both sides of the above equation by $\tau(G)$ we get

$$\begin{aligned}\frac{\tau_{ij}}{\tau(G)} &= \frac{\tau_{i,n}}{\tau(G)} + \frac{\tau_{j,n}}{\tau(G)} - \frac{2\tau_{ij,n}}{\tau(G)}\\ r_{ij} &= r_{i,n} + r_{j,n} - 2z_{ij}\end{aligned} \tag{31}$$

Since each $r_{j,n}$ appears $n-1$ times on the right-hand side of the sum $\sum_{i,k>i} r_{i,k}$, we get

$$\begin{aligned}\sum_{i,k} r_{i,k} &= (n-1)\sum_{j=1}^{n-1} r_{j,n} - 2\sum_{i,k} z_{ik}\\ &= (n-1)\sum_{j=1}^{n-1} r_{j,n} + \sum_{j=1}^{n-1} r_{j,n}\\ &\quad - \left(\sum_{j=1}^{n-1} r_{j,n} + 2\sum_{i,k} z_{ik}\right)\\ KI\,(G) &= n\sum_{j=1}^{n-1} r_{j,n} - \left(\sum_{j=1}^{n-1} r_{j,n} + 2\sum_{i,k} z_{ik}\right)\end{aligned}$$

The above is the same as

$$KI(G) = n\sum_{i=1}^{n-1} z_{ii} - \left(\sum_{i,k} z_{ii}\right) \tag{32}$$

■

See [28] for another other proofs of Theorem 8 starting from the pseudo-inverse $L^{+}(G)$.

VI. Cutset Laplacian Matrix of a Graph and Kirchhoff Index

Recall that the node-conductance matrix Y, also called the reduced Laplacian matrix, is given by

$$Y = AWA^{t} \tag{33}$$

where A is the reduced incidence matrix of G with respect to a specified reference vertex and W is the diagonal matrix of conductances of the elements of G.

Since each row of A represents a cut vector (set of edges incident on a node), we can generalize the notion of Laplacian matrix using fundamental cutset Q_f in place of A.

A. Cutset Laplacian Matrix

Let T be a spanning tree of a connected graph G and Q_f be the fundamental cutset matrix of G with respect to T. If W is the diagonal matrix of edge conductances of G, then the ***cutset Laplacian matrix*** Y_t of G is defined by

$$Y_t = Q_f W Q_f^{t} \tag{34}$$

The matrix Y_t is also called the *short-circuit conductance matrix* of a multiport resistance network, as viewed from the branches of T (called ports). The matrix $Z_t = Y_t^{-1}$ is called the *open-circuit resistance matrix* of the multiport network.

Each diagonal entry of Z_t is the resistance r_{ij} across the nodes i and j of the corresponding defining branch of T.

For example, assuming that all the edge weights are one, the cutset Laplacian matrix Y_t of the connected graph of Figure 2(a) with respect to the tree T in Figure 2(b) is given by

$$\begin{aligned}Y_t = &\begin{bmatrix}1&0&0&0&0&-1&0\\0&1&0&0&0&-1&-1\\0&0&1&0&0&1&0\\0&0&0&1&0&0&1\\0&0&0&0&1&0&0\end{bmatrix}\\ &\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix}\\ &\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\-1&-1&1&0&0\\0&-1&0&1&0\end{bmatrix}\\ = &\begin{bmatrix}2&1&-1&0&0\\1&3&-1&-1&0\\-1&-1&2&0&0\\0&-1&0&2&0\\0&0&0&0&1\end{bmatrix}\end{aligned}$$

So, we get,

$$Z_t = Y_t^{-1} = \begin{bmatrix}8/11&-2/11&3/11&-1/11&0\\-2/11&6/11&2/11&3/11&0\\3/11&2/11&8/11&1/11&0\\-1/11&3/11&1/11&7/11&0\\0&0&0&0&1\end{bmatrix}$$

Note that, the (1, 1) entry of above matrix Z_t is the resistance r_{14} because it corresponds to edge (1, 4). Also, element $z_{ij} = V_i$, where V_i is the voltage across the ith branch of T when a current source of unit value is connected across the nodes of the jth branch of T, as shown in Figure 5.

B. Computing Kirchhoff Index: A Matrix Transformation Approach

In section V we presented a formula to compute the Kirchhoff index using the elements of $Z = Y^{-1}$, where Y is a reduced Laplacian matrix. Since Y is obtained using a star-tree we shall henceforth denote Y by Y_n. In this section,

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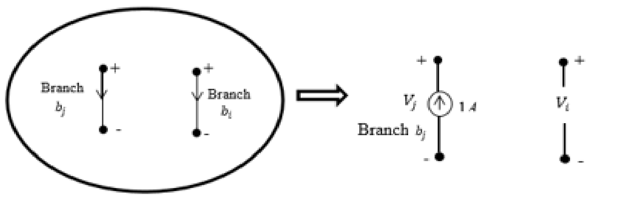

Fig. 5. Voltage V_i across the ith branch when a current source of $1A$ is connected across the nodes of the jth branch.

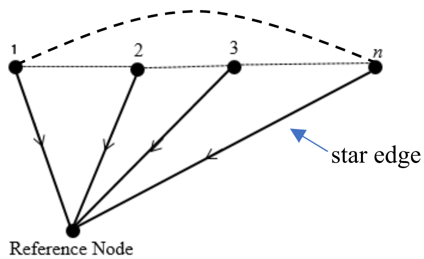

Fig. 6. Star tree T_n.

Fig. 7. Voltage across a branch of T and current injected through the branch.

we present a method to compute the Kirchhoff index from Z_t using a matrix transformation approach.

Note that in view of our definition of the cutset Laplacian, Y_n may be viewed as the cutset Laplacian matrix with respect to the star tree T_n (see Figure 6).

The matrix $(Y_t)^{-1} = (Q_f W Q_f^t)^{-1}$ specifies the relationship between the voltages across the branches of T and the currents injected through these branches (see Figure 7).

That is,

$$V_t = Z_t I_t \tag{35}$$

If Y_n is the Laplacian matrix when the star tree is used, then

$$V_n = Z_n I_n \tag{36}$$

where $Z_n = Y_n^{-1}$.

If Z_n is known, we can find the Kirchhoff index using the formula in Theorem 8. So, given Z_t, our interest is to determine Z_n using a matrix transformation approach. We can then apply (32) on Z_n to compute the Kirchhoff index.

Now we show how to relate Z_n with Z_t.

Theorem 9: Let T be any spanning tree of a graph and T_n be a star tree. Let B_f be the fundamental circuit matrix of the graph $T \cup T_n$ with respect to T and B_{ft} the submatrix of B_f corresponding to the branches of T_n. Then

$$Z_n = B_{ft} Z_t B_{ft}^t$$

Proof: Let Q_{ft} and B_f be the fundamental cutset and fundamental circuit matrices of the graph $T \cup T_n$ with respect to the tree T, and A be the reduced incidence matrix of $T \cup T_n$ with node n as reference.

Then

$$\begin{aligned} Q_f &= [U \quad Q_{fc}] \\ B_f &= [B_{ft} \quad U] \end{aligned}$$

and

$$A = [A_{11} \quad U]$$

where the columns of the first submatrix correspond to the branches of T and the columns of the second submatrix correspond to the branches of T_n.

Note that each row of the reduced incidence matrix A represents a cut. So, the rows of A represent $n-1$ linearly independent cutsets. This means that each row of Q_f can be written as a linear combination of the rows of A. That is,

$$Q_f = \left[U \,|\, Q_{fc}\right] = A_{11}^{-1}[A_{11}U]$$

So

$$Q_{fc} = A_{11}^{-1}$$

and by equation (14)

$$B_{ft} = -Q_{fc}^t = -(A_{11}^{-1})^t$$

Now

$$\begin{aligned} Y_t &= Q_f W Q_f^t \\ &= A_{11}^{-1} A W (A_{11}^{-1} A)^t \\ &= A_{11}^{-1} (A W A^t)(A_{11}^{-1})^t \\ &= A_{11}^{-1} Y_n (A_{11}^{-1})^t \end{aligned}$$

So

$$\begin{aligned} Z_t &= Y_t^{-1} \\ &= A_{11}^t Y_n^{-1} A_{11} \\ &= A_{11}^t Z_n A_{11} \end{aligned}$$

and

$$\begin{aligned} Z_n &= \left(A_{11}^{-1}\right)^t Z_t \left(A_{11}^{-1}\right). \\ &= B_{ft} Z_t B_{ft}^t \end{aligned} \tag{37}$$

■

As an example, for the graph in Figure 2(a), let the datum node is v_2. We get the following reduced Laplacian matrix by removing the 2nd row and 2nd column from the Laplacian matrix of the graph. Again, we have assumed that all the edge weights are unity. Then

$$Y_n = L(\bar{z}) = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

$$Z_n = Y_n^{-1} = \begin{bmatrix} 0.73 & 0.27 & 0.45 & 0.36 & 0.36 \\ 0.27 & 0.72 & 0.55 & 0.64 & 0.64 \\ 0.45 & 0.55 & 0.91 & 0.73 & 0.73 \\ 0.36 & 0.64 & 0.73 & 1.18 & 1.18 \\ 0.36 & 0.64 & 0.73 & 1.18 & 2.18 \end{bmatrix},$$

and using Theorem 9

$$Z_n = B_{ft} Z_t B_{ft}^t = \begin{bmatrix} 0.73 & 0.27 & 0.45 & 0.36 & 0.36 \\ 0.27 & 0.72 & 0.55 & 0.64 & 0.64 \\ 0.45 & 0.55 & 0.91 & 0.73 & 0.73 \\ 0.36 & 0.64 & 0.73 & 1.18 & 1.18 \\ 0.36 & 0.64 & 0.73 & 1.18 & 2.18 \end{bmatrix}.$$

Using (32), Kirchhoff Index of G is calculated as

$$KI(G) = 16.8.$$

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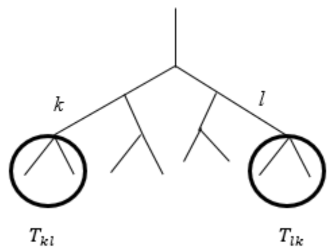

Fig. 8. Definition of $\boldsymbol{T_{kl}}$ and $\boldsymbol{T_{lk}}$.

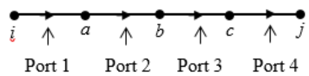

Fig. 9. Path from vertex i to vertex j.

C. Kirchhoff Polynomial of a Graph and a Formula for Kirchhoff Index

In this section, we determine a formula for the Kirchhoff index in terms of the elements of Z_t. We define a new concept called the *Kirchhoff polynomial* of a graph. This is a generalization of the formula in (32) for $KI(G)$ given in terms of the elements of $Z_n = (Y_n)^{-1}$, where Y_n is the reduced Laplacian matrix of the graph.

Let Y_t be the cutset Laplacian matrix of a resistance network G with respect to a spanning tree T. Let $Z_t = (Y_t)^{-1} = [z_{ij}]$. *Kirchhoff polynomial* of G is a polynomial $\sum_{i,j} c_{ij} z_{ij}$ that expresses Kirchhoff index of G in terms of the elements of Z_t. That is,

$$KI(G) = \sum c_{ij} z_{ij}. \tag{38}$$

To present our next result we need to introduce some new notation

Consider a complete graph K_n. Let T be a spanning tree of K_n. Consider any two ports k and l from T.

Suppose we remove k from T, then T will be disconnected into two trees. One of them will not contain l. We shall call this tree as T_{kl}. Similarly, if we remove port l from T, then the tree that does not contain k will be denoted by T_{lk}. $|T_{kl}|$ and $|T_{lk}|$ will denote the number of vertices in T_{kl} and T_{lk} respectively. See Figure 8.

Theorem 10: Given a graph G with weight matrix$W$. Let T be a spanning tree of$G$.Let $Z_t = [z_{ij}]$ be the open-circuit resistance matrix with respect to T.Then the Kirchhoff Index *KI(G)* is given by

$$\begin{aligned} KI(G) &= \sum\nolimits_k c_{kk} z_{kk} + \sum\nolimits_{k,l>k} c_{kl} z_{kl} \\ &= \sum\nolimits_k |T_k^{(1)}| \cdot |T_k^{(2)}| + 2 \sum_{\substack{kl \\ k>l}} \pm |T_{kl}| \cdot |T_{lk}|. \end{aligned} \tag{39}$$

Proof: We first determine a formula for each r_{ij}. Consider the path from vertex i to vertex j in the spanning tree T. To illustrate the ideas in our development, let this path be as given in Figure 9.

For convenience, in Figure 9 the ports are oriented similarly. But in general, the ports can be oriented arbitrarily.

Consider now the 3-node equivalent representation of the graph shown in Figure 10. This network can be obtained by repeated star-delta transformation at the remaining nodes.

Then by equation (31)

$$r_{bj} = r_{bc} + r_{jc} - 2V_{jc} = r_{bc} + r_{jc} + r_{bc} + 2z_{34}$$

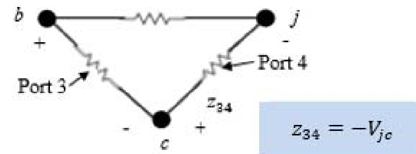

Fig. 10. 3-node equivalent representation of the graph given in Figure 9.

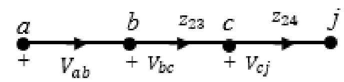

Fig. 11. Computing r_{aj}.

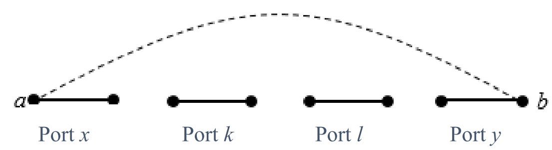

Fig. 12. Computing $\boldsymbol{C_{kl}}$.

Note that, if port 4 is oriented from j to c, then

$$r_{bj} = r_{bc} + r_{cj} - 2z_{34}$$

as in equation (31).

Next consider r_{aj}, as shown in Figure 11,

$$\begin{aligned} r_{aj} &= r_{ab} + r_{bj} - 2V_{jb} \\ &= r_{ab} + r_{bj} + 2(z_{23} + z_{24}) \end{aligned}$$

In the above we have replaced V_{jb} by $-z_{23} - z_{24}$.

$$r_{aj} = r_{ab} + r_{bc} + r_{cj} + 2(z_{23} + z_{24}) + 2z_{34}.$$

Continuing

$$\begin{aligned} r_{ij} = (r_{ia} + r_{ab} + r_{bc} + r_{cj}) + 2(z_{12} + z_{13} + z_{14}) \\ + 2(z_{23} + z_{24}) + 2z_{34}. \end{aligned}$$

Note that resistances r_{ia}, r_{ab}, r_{bc} and r_{cj} are diagonal elements of Z_t. For instance, r_{ab} is the diagonal element z_{22}.

From the above we can see that the transfer resistance, say z_{24} appears exactly once as $2z_{24}$ in the expressions of each of the resistance distances r_{ij}, r_{aj} and r_{bj}. Generalizing this we can state that each z_{kl} appears exactly once as $2z_{kl}$ in each r_{xy} when the unique path in T containing ports x and y spans ports k and l as shown in Figure 12. Similarly, each element z_{ii} appears exactly once in each r_{xy} when the unique path from x-to-y in T spans port i.

Summarizing, if K_n is the complete graph on the vertices of T then

$$\begin{aligned} c_{kl} &= 2\,(\#\text{number of edges in } K_n \text{ that span ports } k \text{ and } l), \\ &\quad \text{if } k \text{ and } l \text{ are similarly oriented} \\ &= -2\,(\#\text{number of edges in } K_n \text{ that span ports } k \text{ and } l), \\ &\quad \text{otherwise.} \\ &= 2\,|T_{kl}| \cdot |T_{lk}|. \end{aligned}$$

and

$c_{kk} = \#$ number of edges of K_n in the fundamental cutset defined by port k.

$$= \left|T_k^{(1)}\right| \cdot \left|T_k^{(2)}\right|$$

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where $T_k^{(1)}$ and $T_k^{(2)}$ are the two trees that result when port k is removed from the tree.

This proves the result. ■

In the case when T is a star tree

$$\begin{aligned} T_k^{(1)} &= 1 \text{ for all } k \\ T_k^{(2)} &= n-1 \text{ for all } k \\ |T_{kl}| &= 1 \\ |T_{lk}| &= 1 \end{aligned}$$

So, in this case

$$\begin{aligned} c_{kk} &= n-1 \\ c_{kl} &= -2, \quad k \neq l, \quad \text{because all ports are dissimilarly oriented.} \end{aligned}$$

and

$$\begin{aligned} KI(G) &= (n-1)\,Tr(Z_t) - 2\sum_{k>l} z_{kl} \\ &= nTr(Z_t) - \sum_{k,l} z_{kl} \end{aligned} \tag{40}$$

This verifies the formula in Theorem 8 for the Kirchhoff index when the star tree is used in defining the cutset Laplacian matrix. Thus Theorem 10 is a generalization of Theorem 8.

As an example, for the graph given in Figure 2(a), Z_t is given in section VI.A and the port numbers for the tree in Figure 2(b) are

Edge $e_2 \to$ Port 1, Edge $e_3 \to$ Port 2, Edge $e_4 \to$ Port 3, Edge $e_5 \to$ Port 4, Edge $e_7 \to$ Port 5.

c_{ij}'s are

$$\begin{aligned} &c_{11} = 5, \quad c_{12} = 2, \quad c_{13} = -1, \quad c_{14} = 2, \quad c_{15} = 1, \\ &c_{21} = 2, \quad c_{22} = 8, \quad c_{23} = -4, \quad c_{24} = -4, \quad c_{25} = -2, \\ &c_{31} = -1, \quad c_{32} = -4, \quad c_{33} = 5, \quad c_{34} = 2, \quad c_{35} = 1, \\ &c_{41} = 2, \quad c_{42} = -4, \quad c_{43} = 2, \quad c_{44} = 8, \quad c_{45} = 4, \\ &c_{51} = 1, \quad c_{52} = -2, \quad c_{53} = 1, \quad c_{54} = 4, \quad c_{55} = 5, \end{aligned}$$

Using (39), we get Kirchhoff Index $KI(G) = 16.8$.

VII. Weighted Kirchhoff Index of a Graph

In this section, we define the concept of the weighted Kirchhoff Index, generalizing the concept of Kirchhoff Index.

Consider a weighted undirected graph G with each edge (i, j) assigned weight w_{ij}. Treating w_{ij}'s as conductances of a resistance network, let r_{ij} denote the resistance distance between nodes i and j. Suppose we associate a weight w_{ij}^* with each r_{ij}. Then the *weighted Kirchhoff Index* $WKI(G)$ of G is defined as [31]

$$WKI(G) = \sum_{i>j} w_{ij}^* r_{ij}.$$

Now we wish to determine a formula to compute $WKI(G)$.

Let T be a spanning tree of the given graph G and Y_t the cutset Laplacian matrix $Q_f W Q_f^t$ where Q_f is the fundamental cutset matrix of G with respect to T.

Let $Z_t = [z_{ij}] = Y_t^{-1}$ be the open-circuit resistance matrix. We wish to express $WKI(G)$ as

$$WKI(G) = \sum c_{ij}^* z_{ij}.$$

In the following the direct product $\oplus$ of two matrices X and Y will be defined as

$$X \oplus Y = \sum x_{ij} y_{ij}$$

Theorem 11:

$$\begin{aligned} WKI(G) &= \sum c_{ij}^* z_{ij} \\ &= C^* \oplus Z_t \end{aligned}$$

where Z_t is the open-circuit resistance matrix of G with respect to the spanning tree T and $C^* = Q_f^* W^* Q_f^{*t}$ where Q_f^* is the fundamental cutset matrix of the complete graph K_n with respect to T. Note that Z_t is defined using weights W on the edges of G and C^* is defined using weights W^* on the edges of K_n.

Proof: To determine c_{ij}^*, consider the elements c_{ij} used in the formula for $KI(G)$ given in Theorem 10. We can see that c_{ij}'s are the elements of the matrix

$$C = Q_f^* Q_f^{*t}$$

where Q_f^* is the fundamental cutset matrix of the complete graph K_n on the vertices of T. Following the same development as in section VI that led to the definition of c_{kl}, we can see that the c_{ij}^*'s are the elements of the matrix

$$C^* = Q_f^* W^* Q_f^{*t}$$

where W^* is the diagonal matrix of w_{ij}^*'s associated with r_{ij}'s.

Then we can write

$$\begin{aligned} WKI(G) &= \sum c_{ij}^* r_{ij} \\ &= C^* \oplus Z_t \end{aligned}$$

This proves the theorem. ■

If $w_{ij}^* = 1$ for all i and j then $WKI(G)$ becomes the same as the Kirchhoff Index. See equation (39) .

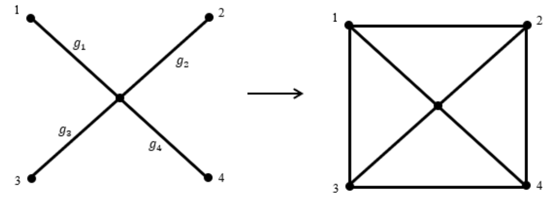
Let D be a subset of pairs of $V \times V$. Then the sum of the r_{ij}'s where $(i, j) \in D$, can be obtained from Theorem 11 by setting $w_{ij}^* = 0$ for all entries of W^* corresponding to the pairs (i, j) that are not in D. So,

Corollary 12: $\sum_{D \leq V \times V} w_{ij}^* r_{ij} = C^* \oplus Z$ where $C^* = Q_f^* W^* Q_f^{*t}$ with the diagonal entry $w_{ij}^* = 0$ for $(i, j) \notin D$. ■

One can also see that Theorem 11 and Corollary 12 generalize and unify the results of both theorems Theorem 8 and Theorem 9.

VIII. Weighted Kirchhoff Index of a Graph and Generalization of Foster's Theorems

In 1949, Foster [17] proved a theorem called Foster's first theorem. This theorem gives an identity involving the sum of the resistance distances. A graph-theoretic proof of this theorem was given in [29]. In [18] Foster gave a generalization of his first theorem. In [20] Tetali proved this theorem using certain results from the theory of Markov Chains. Building on Tetali's probabilistic approach, Palacios gave another proof of Foster's second theorem [21]a, [23]. In these papers, Palacios also gave an extension of Foster's second theorem. In 2007, Cinkir [24] gave a generalization of all of Foster's theorems. Connections between resistance distances and random walks on graph have been discussed in several works. See [13] and [14] for examples. See [19] for the application of random



$$g_{24} = \frac{g_2 g_4}{g_1 + g_2 + g_3 + g_4}$$

Fig. 13. Star-delta transformation.

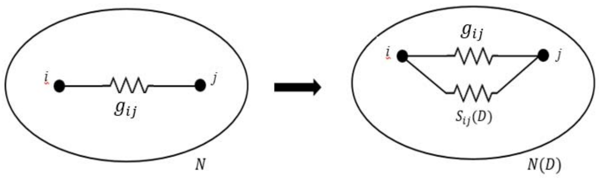


Fig. 14. Multiple star-delta transformations.

walk and Foster's theorem in the analysis of on-line algorithms. See [10]–[12] for connection of electrical circuit theory to topics in medical statistics and combinatorial designs.

In this section, we provide further advances on Foster's theorems.

A. Basic Concepts and Definitions

Consider a network N of positive resistances. Let V be the set of nodes in N. Let n denote the number of nodes in N. We assume that the nodes are numbered $1, 2, \ldots, n$. So $V = \{1, 2, ..n\}$. Let g_{ij} be the value of the conductance of the resistance element connecting nodes i and j. Let r_{ij} denote the resistance distance of N across the pair of nodes i and j.

1) Star-Delta Transformation: Consider a node v. Let $g_1, \ldots, g_k$ be the conductances of the edges incident on v, with $1, 2, \ldots, k$ denoting the other end nodes of these edges. Star-delta transformation at v is the operation of removing node v from N and adding a new element (i, j) with conductance $g_i g_j \big/ d(v)$ for all $k \leq i, j \leq k$, where $d(v)$ is the sum of the conductances of the edge (see Figure 13). Let N' be the resulting network.

It is well known in circuit theory that resistance distance across nodes i and j in $N^{'}$ is the same as r_{ij} in N as long as these nodes remain in $N^{'}$ after a star-delta transformation.

2) Multiple Star-Delta Transformations: We wish to note that multiple star-delta transformation discussed below is also known as *Kron-reduction* [30].

Let D be a proper subset of nodes of N, that is, $D \subset V$. Suppose we perform star-delta transformations successively at the nodes in D, one at a time. Let $N(D)$ denote the resulting network. Clearly $N(D)$ has $n-k$ nodes when $k = |D|$. At the end of the multiple star-delta transformations, a new resistance element connecting i and j will be created in $N(D)$. Let the conductance value of the new element be $S_{ij}(D)$. Thus, the total value of the conductance of the element connecting i and j in $N(D)$ will be $y_{ij} + S_{ij}(D)$. See Figure 14.

Let

$$s_{ij}\,(k) = \sum_{\substack{D \subset V \\ |D| = k}} S_{ij}\,(D). \tag{41}$$

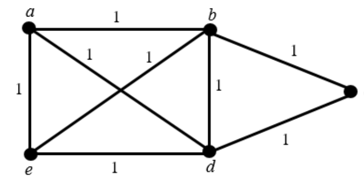


Fig. 15. A 5-node resistance network N.

That is, $s_{ij}\,(k)$ is the sum of $S_{ij}\,(D)$'s for all subsets of nodes of size k.

As an example, consider a 5-node resistance network N given in Figure 15. For this, there are ten 2-element subsets of nodes. These subsets are:

$$\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{b, e\}, \{b, d\}, \{b, e\}, \\ \{c, d\}, \{c, e\}, \{d, e\}.$$

For each subset D of nodes, the corresponding network $N\,(D)$ is shown in Figure 16. In this figure, dotted edges indicate the new resistance elements along with the corresponding $S_{ij}\,(D)$'s.

Then, using (41) we have

$$s_{ab}\,(2) = \frac{3}{7} + \frac{1}{3} + 1 = \frac{37}{21}, \quad s_{ac}\,(2) = \frac{2}{3} + \frac{4}{11} + \frac{4}{11} = \frac{46}{33},$$
$$s_{ad}\,(2) = \frac{3}{7} + \frac{9}{11} + \frac{1}{3} = \frac{365}{231}, \quad s_{ae}\,(2) = \frac{2}{7} + \frac{2}{3} + \frac{2}{7} = \frac{26}{21},$$
$$s_{bc}\,(2) = \frac{4}{11} + \frac{4}{11} = \frac{8}{11}, \quad s_{bd}\,(2) = \frac{5}{6} + 1 + \frac{5}{6} = \frac{8}{3},$$
$$s_{be}\,(2) = \frac{1}{3} + \frac{9}{11} + \frac{3}{7} = \frac{365}{231}, \quad s_{cd}\,(2) = \frac{4}{11} + \frac{4}{11} = \frac{8}{11},$$
$$s_{ce}(2) = \frac{4}{11} + \frac{4}{11} + \frac{2}{3} = \frac{46}{33}, \quad s_{de}(2) = \frac{9}{11} + \frac{1}{3} + \frac{3}{7} = \frac{365}{231}$$

B. Foster's Theorems

1) Foster's First Theorem: Consider a resistance network N. Let N have n nodes and m elements $e_1, e_2, \ldots, e_m$. The conductance of each e_i will be denoted by g_{i_1,i_2}. Also, the two nodes of each e_i will be denoted by i_1 and i_2. If r_{i_1,i_2} denotes the resistance distance across the pair of nodes i_1 and i_2, then we have the following theorem due to Foster [17]. See [29] for a proof of this theorem.

Theorem 12 (Foster's First Theorem):

$$\sum_{i=1}^{m} g_{i_1,i_2} r_{i_1,i_2} = n - 1 \tag{42}$$

2) Foster's Second Theorem: Foster's second theorem is based on the operation of star-delta transformation at a single node.

Consider a node v. Let $g_1, \ldots, g_k$ be the conductances of the edges incident on v, with $1, 2, \ldots, k$ denoting the other end nodes of these edges. Recall that star-delta transformation at v removes node v from N and adds a new element (i, j) with conductance $g_i g_j \big/ d(v)$ for all $k \leq i, j \leq k$.

The following theorem is by Foster [18].

Theorem 13 (Foster's Second Theorem):

$$\sum_{v} \sum_{i<j} r_{ij} \frac{g_i g_j}{d(v)} = n - 2 \tag{43}$$

where the sum is extended over all pairs of adjacent elements incident on a common node v. ■

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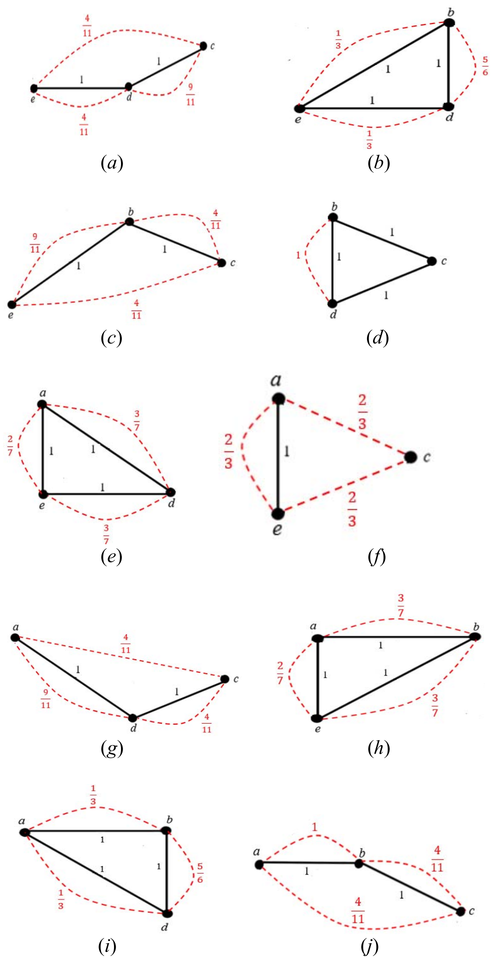

Fig. 16. Network $N(D)$ for each subset D of nodes. (a) Star-Delta transformation. (b) Star-Delta transformation at nodes $\{a, b\}$ at nodes $\{a, c\}$. (c) Star-Delta transformation (d) Star-Delta transformation at nodes $\{a, d\}$ at nodes $\{a, e\}$. (e) Star-Delta transformation (f) Star-Delta transformation at nodes $\{b, c\}$ at nodes $\{b, d\}$. (g) Star-Delta transformation (h) Star-Delta transformation at nodes $\{b, e\}$ at nodes $\{c, d\}$. (i) Star-Delta transformation (j) Star-Delta transformation at nodes $\{c, e\}$ at nodes $\{d, e\}$.

Next, we present Foster's two theorems using the concept of weighted Kirchhoff index and choosing w_{ij}^*'s appropriately.

3) Foster's first theorem using weighted Kirchhoff index: Theorem 14: If $w_{ij} = g_{ij}$ then

$$WKI(N) = \sum_{i<j} g_{ij} r_{ij} = n - 1.$$

Proof: Note that $g_{ij} = 0$ if there is no resistance element connecting i and j. So, in that case, we get the original statement of Foster's theorem, namely,

$$\sum_{i \sim j} g_{ij} r_{ij} = n - 1.$$

Here $i \sim j$ means there is an element connecting i and j. So we get the result in the theorem. ■

4) Foster's Second Theorem Using Weighted Kirchhoff Index: Theorem 15: If $w_{ij} = s_{ij}(1)$ then

$$WKI(N) = \sum_{i<j} s_{ij}(1) r_{ij} = n - 2.$$

■

In fact, Foster's theorems can be represented as in Theorem 11 by defining W^* appropriately. That is, we can get alternate forms of Foster's theorems in terms of the elements z_{ij} of the open-circuit resistance matrix Z.

We next state and prove a generalization of Foster's theorems.

5) Generalized Foster's theorem: Theorem 16: If $w_{ij} = s_{ij}(k), k \geq 1$ then

$$WKI(N) = \sum_{i<j} s_{ij}(k) r_{ij} = (n-k-1)\binom{n-1}{k-1}$$

Proof: Consider a resistance network N of n nodes with nodes numbered $1, 2, \ldots, n$. Let $V = \{1, 2, \ldots, n\}$. Let D be a proper subset of V and $|D| = k$. Then the network $N(D)$ that results after *Star-Delta Transformations* at the nodes of D will have $n - k$ nodes. So, applying Foster's first theorem on $N(D)$, we get

$$\sum_{i<j} (g_{ij} + S_{ij}(D)) r_{ij} = n - k - 1. \tag{44}$$

Equation (44) can be rewritten as

$$\sum_{i<j} S_{ij}(D) r_{ij} + \sum_{i<j} g_{ij} r_{ij} = n - k - 1. \tag{45}$$

Let us now write similar equations for all the $\binom{n}{k}$ subsets of V of size k and sum up both the right-hand side and left-hand side terms.

Then we get

$$\sum_{D \subset V} \sum_{i<j} S_{ij}(D) r_{ij} + \sum_{D \subset V} \sum_{i<j} g_{ij} r_{ij} = \binom{n}{k}(n-k-1). \tag{46}$$

If $|D| = k$, then equation (46) can be rewritten as

$$\sum_{i<j} s_{ij}(k) r_{ij} + \sum_{D \subset V} \sum_{i<j} g_{ij} r_{ij} = \binom{n}{k}(n-k-1). \tag{47}$$

Consider the second term $\sum_{D \subset V} \sum_{i<j} g_{ij} r_{ij}$ in (46). In this summation, $g_{ij} r_{ij}$ will be present only if D does not contain i or j. There are $\binom{n-2}{k}$ subsets of V that satisfy this requirement. In all other cases, $g_{ij} r_{ij}$ will not be present. Thus, each term $g_{ij} r_{ij}$ appears exactly $\binom{n-2}{k}$ times in the second sum in (46). So, we can rewrite (46) as

$$\sum_{i<j} s_{ij}(k) r_{ij} + \binom{n-2}{k} \sum_{i<j} g_{ij} r_{ij} = \binom{n}{k}(n-k-1).$$

That is, $\sum_{i<j} s_{ij}(k) r_{ij} + \binom{n-2}{k}(n-1) = \binom{n}{k}(n-k-1)$, by Theorem 14.

So,

$$\begin{aligned}
\sum_{i<j} s_{ij}(k) r_{ij} &= (n-k-1)\binom{n}{k} - \binom{n-2}{k}(n-1) \\
&= (n-k-1)\left[\binom{n}{k} - \frac{(n-1)}{(n-k-1)}\binom{n-2}{k}\right] \\
&= (n-k-1)\left[\binom{n}{k} - \frac{(n-1)!}{k!\,(n-k-1)!}\right] \\
&= (n-k-1)\left[\binom{n}{k} - \binom{n-1}{k}\right] \\
&= (n-k-1)\left[\binom{n-1}{k-1}\right],
\end{aligned}$$

where the identity $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ is used. ■

The above theorem is from [31].

For example, the $WKI(N)$ of the 5-node resistance network N (Figure 17) for $k = 2$ is calculated below. Note that $|D| = 2$. The resistance distance r_{ij} for each pair of nodes for this network N is

$$\begin{aligned} &r_{ab} = 0.475, \quad r_{ac} = 0.875, \quad r_{ad} = 0.475, \quad r_{ae} = 0.500,\\ &r_{bc} = 0.600, \quad r_{bd} = 0.400, r_{be} = 0.475, \quad r_{cd} = 0.600,\\ &r_{ce} = 0.875, \quad r_{de} = 0.475. \end{aligned}$$

By using the above calculated r_{ij}'s and the $s_{ij}(2)$'s calculated earlier, we can calculate $r_{ij}s_{ij}(2)$ for each pair of nodes as given below:

$$\begin{aligned} r_{ab}s_{ab}(2) &= 0.837, \quad r_{ac}s_{ac}(2) = 1.219,\\ r_{ad}s_{ad}(2) &= 0.750, \quad r_{ae}s_{ae}(2) = 0.619,\\ r_{bc}s_{bc}(2) &= 0.436, \quad r_{bd}s_{bd}(2) = 1.066,\\ r_{be}s_{be}(2) &= 0.750, \quad r_{cd}s_{cd}(2) = 0.436,\\ r_{ce}s_{ce}(2) &= 1.219, \quad r_{de}s_{de}(2) = 0.750, \end{aligned}$$

So, $WKI(N) = \sum_{i<j} s_{ij}(k) r_{ij} = 8.08 \cong 8$.

For $n = 5$ and $k = 2$, we have

$$\sum_{i<j} s_{ij}(k) r_{ij} = (n-k-1)\binom{n-1}{k-1} = 3\binom{4}{1} = 8,$$

verifying Theorem 16.

C. Dual Form of Foster's First Theorem

Circuits and cutsets are dual concepts [25]. Similarly rank and duality are dual concepts. In this subsection we address the question whether one could assign weights appropriately so that the corresponding weighted Kirchhoff index is equal to $m - n + 1$, the nullity. We shall answer this question in the affirmative.

Note that the largest value that k can take in Theorem 16 is equal to $n - 2$, since at least two nodes are needed to define resistance distance.

Theorem 17 (Dual of Foster's First Theorem): $\sum_{\substack{i<j\\ i\sim j}} s_{ij}(n-2) r_{ij} = m - n + 1 =$ nullity of graph G.

Proof: Since

$$\left(s_{ij}(n-2) + g_{ij}\right) r_{ij} = 1$$

we have

$$\sum_{i\sim j} (s_{ij}(n-2) + g_{ij}) r_{ij} = m$$

We can rewrite the above as

$$\sum_{i\sim j} (s_{ij}(n-2) r_{ij} + \sum_{i\sim j} g_{ij} r_{ij} = m$$

Then using Theorem 14 we get

$$\begin{aligned} \sum_{i\sim j} (s_{ij}(n-2) r_{ij} &= m - n + 1\\ &= \text{nullity of } G. \end{aligned}$$

■

IX. Summary

In this paper, we have studied two structural metrics of a weighted connected graph, namely the resistance distance and the Kirchhoff Index. We have presented several new results that extend, generalize, and unify earlier contributions reported in the literature. Given a connected weighted graph G, our main contributions are summarized below.

- Given the reduced Laplacian matrix Y of a graph G (same as the node to conductance matrix of the corresponding resistance network) and its inverse Z, a formula to compute the Kirchhoff Index in terms of the elements of Z is given. (Theorem 8).
- The concept of cutset Laplacian matrix Y_t of a graph G with tree T is given. This generalize the concept of reduced Laplacian matrix Y_n.
- Given the cutset Laplacian matrix Y_t of a graph G and its inverse Z_t, a method to compute the Kirchhoff Index of G is given, using a matrix transformation approach. (Theorem 9). In this approach it is shown how to convert Z_t to Z_n and then use Theorem 8 to compute Kirchhoff Index.
- Given the cutset Laplacian matrix Y_t of a graph G and its inverse Z_t, a formula to compute the Kirchhoff Index of G is given. For this purpose, the concept of Kirchhoff polynomial $\sum c_{ij} z_{ij}$ where z_{ij} are elements of Z_t is first defined. It is shown that the Kirchhoff Index of G can be expressed as $\sum c^*_{ij} z_{ij}$, where c^*_{ij} are elements of the matrix $Q^*_f Q^{*t}_f$ and Q^*_f is the fundamental cutset matrix of a complete graph with respect to the tree T (Theorem 10). This generalizes the result in Theorem 8. This formula is applicable even when the Kirchhoff Index is restricted to a subset of resistance distances.
- The concept of weighted Kirchhoff Index of G is defined. A formula to compute the weighted Kirchhoff Index is given using the result in Theorem 10 (Theorem 11). This unifies the results in Theorems 8 and 10.
- Foster's two theorems are shown to be invariants of the weighted Kirchhoff Index for special cases of weights. (Theorems 14 and 15).
- A generalization of Foster's theorem that retains the circuit theoretic elegance of Foster's original theorems is given (Theorem 16). Unlike the other generalization reported in [23] and [24] our generalization results in an invariant that is independent of the edge weights of the given network.
- Alternate forms of Foster's theorems in terms of the elements of Z_t, (instead of the resistance distances used in Foster's original theorems) can be obtained using the formulas in Theorems 8, 10 and 11.
- The dual form of Foster's theorem is given (Theorem 17). This shows how the dual can be obtained using the primal (Foster's First Theorem) and certain basic properties of resistance distances. Also, Theorem 16 on generalized Foster's theorem contains both the primal and dual as extreme special cases.

The research findings in this paper provide the theoretic foundation to study several issues of interest in network science.

We suggest two problems for further investigations. Results in [13]–[15] suggest that end-to-end or link level congestion in a communication network can be controlled by selecting the link weights (conductances) appropriately. This is equivalent to the problem of designing a resistance network such that the resistance distance between specified pair of nodes are within prescribed levels. Another problem is that of partitioning a network into clusters so that the Kirchhoff Index of each cluster and the inter-cluster Kirchhoff Index of each pair of clusters are within specified limits. Such a clustering will help in devising protection schemes to contain failure cascades in power and other networks.

Resistance networks had been studied extensively in the 1960's and 1970's. For example, see [32] and [33]. These results can be of use in developing mathematical programming formulations for the above two problems.

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