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Fault tolerance of hypercube like networks: Spanning laceability under edge faults ☆,☆☆,☆☆☆

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A B S T R A C T

Given two vertices u and v in a connected undirected graph G, a $w*$-container $C(u,v)$ is a set of w internally vertex disjoint paths between u and v spanning all the vertices in G. A bipartite graph G is w^*-laceable if there exists a w^*-container between any two vertices belonging to different partitions of G. In [8], [33] a class B'_n of bipartite graphs called hypercube-like bipartite networks was defined. In [22], Lin et al. showed that every graph in B'_n is w^*-laceable for every $1 \leq w \leq n$. We define a graph is f-edge fault tolerant w^*-laceable if $G - F$ is w^*-laceable for any arbitrary subset F of edges of G with $|F| \leq f$. In this paper we show that every graph in B'_n is f-edge-fault tolerant w^*-laceable for every $0 \leq f \leq n-2$ and $1 \leq w \leq n-f$ which generalize Lin's result. We also give generalization of two other results in [22,27].

1. Introduction

In the last three or four decades, we have witnessed enormous advances in semiconductor technology. These advances have resulted in the production of large systems composed of interconnections of small components with high probability of failure. Such failures lead to increasing probability of failure of the overall system, causing disruption of service. Thus fault tolerance has come to play a central role in the design and analysis of large scale systems encountered in the modern era. In general, there are two approaches to achieve fault tolerance: hardware and software (protocols). Both these approaches require the systems/networks under consideration to possess certain topological properties. For example, designing a survivable logical topology routing in an IP-over-WDM optical network requires the existence of disjoint paths between certain specified pairs of vertices in the optical networks [17], [31], [39]. For an approach for fault identification in sensor networks existence of certain codes called identifying codes is required [15], [35], [37]. As example, in [33] a new class of networks called hypercube-like networks was introduced as a choice of interconnection topology for multiprocessor systems. These networks were also subsequently introduced in an independent work [8].

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These research trends have motivated our work in this paper where we study fault tolerance properties of hypercube-like networks measured in terms of the existence of certain disjoint paths after occurrence of edge faults. Our results generalize those properties established earlier for hypercube-like networks without faults [22], [27]. In this section we present basic definitions that will be needed for our development in the rest of this paper, along with a brief review of related works.

Usually, interconnection networks can be represented as graphs in which vertices represent processors and edges represent communication links.

For graph theoretic notations not defined here we follow [2]. Consider a graph $G = (V(G), E(G))$ with $V(G)$ and $E(G)$ denoting the vertex set and edge set of G, respectively. $N(u)$ will denote the set of all neighbors of vertex u in G and $d_G(u) = |N_G(u)|$ will denote the degree of u. G is called k-regular if the degree of every vertex in G is equal to k. For a set $F \subseteq E(G)$, $G - F$ will denote the graph that results after deleting the set of edges in F. The connectivity (respectivity, edge connectivity) of a graph is the minimum number of vertices (respectively, edges) whose removal disconnects G or makes G trivial.

A set of paths between vertices u and v are called internally disjoint paths if they have no common vertices except u and v. A path is hamiltonian if it contains all the vertices of G. A graph is hamiltonian connected if it contains a hamiltonian path between every pair of vertices in G. A cycle in a graph G is a hamiltonian cycle if it contains all the vertices of G. A graph is hamiltonian if it contains a hamiltonian cycle. A bipartite graph is hamiltonian laceable if it contains a hamiltonian path between any two vertices from different partitions of the graph.

In [12], motivated by an application in communication networks, Hsu proposed the concept of container to evaluate the communication reliability of an interconnection network. For $\{u, v\} \subseteq V(G)$, the w-container $C(u, v)$ of G is a set of w internally disjoint paths between u and v.

In this paper, we are concerned with a specific type of container, w^*-container. See [22] for an application of this concept in bioinformatics and neuroinformatics. A w^*-container $C(u, v)$ is a set of w internally disjoint (u, v)-paths which contains all the vertices of G. A graph G is w^*-connected if there exists a w^*-container between any two distinct vertices u and v.

Let G be a bipartite graph with bipartition V_1 and V_2 with $|V_1| \geq |V_2|$. The graph G is w^*-laceable if for each $u \in V_1$, $v \in V_2$ there exists a w^*-container between u and v for some w, $1 \leq w \leq \kappa(G)$. Obviously, the partite sets of any bipartite w^*-laceable graph with $w \geq 2$ have equal size partitions.

The concept of spanning fan and spanning connectivity play an important role in the routing of large-scale networks [13,14,21–24]. Let $u \in V(G)$ and $S = \{v_1, v_2, \cdots, v_k\} \subseteq V(G) \setminus \{u\}$. An (u, S)-fan is a set of internally disjoint paths of G such that for each v_i, there is a path from u to v_i [6]. A spanning fan is a fan that contains all the vertices. The spanning connectivity of a graph G, $\kappa^*(G)$, is the largest integer k such that G is i^*-connected for every i, $1 \leq i \leq k$. The concept of spanning connectivity has been widely investigated. The spanning connectivity of many networks, such as generalized Petersen graphs $P(n, 3)$ [34], WK-recursive networks [38], arrangement graphs [19], tori network [18] and augmented cubes [20] have been studied. Also, Huang and Hsu studied the spanning connectivity of line graph [14]. Lin et al. discussed the relationship between connectivity, spanning connectivity and spanning fan connectivity [23]. Hsu et al. gave the lower bound of the spanning connectivity for networks [21].

Motivated by the high likelihood of link failures in communication networks, we consider edge fault-tolerance ability of networks in this paper. A graph G is f-edge fault-tolerant Hamiltonian if $G - F$ is Hamiltonian for any edge subset F of G with $|F| \leq f$. A graph G is f-edge fault-tolerant Hamiltonian connected if $G - F$ is Hamiltonian connected for any edge subset F of G with $|F| \leq f$. A graph G is f-edge fault-tolerant w^*-connected if $G - F$ is w^*-connected for any arbitrary edge set F with $|F| \leq f$. A bipartite graph G is f-edge fault tolerant hamiltonian laceable if $G - F$ is hamiltonian laceable for any edge subset F of G with $|F| \leq f$. A bipartite graph G is f-edge fault-tolerant w^*-laceable if $G - F$ is w^*-laceable for any arbitrary edge set F with $|F| \leq f$.

The hypercube is the most popular interconnection network used in parallel processing systems. It possesses several attractive properties, see [30] for a detailed introduction to hypercubes and their properties. To overcome some limitations of the hypercube, certain variants such as the crossed cube [7], twisted cube [1], and Mobius cube [5] have been studied. To produce a unified theory of hypercube and their variants, Vaidya, Rao and Shankar [33] introduced the hypercube-like network which are also called bijective connection network (In short, BC networks), by Fan and He [8]. Several fault-tolerance properties of hypercubes and their invariants may be found in [10], [11], [13], [18], [20], [23], [24], [36]. In [26], [28] Park et al. studied the existence of disjoint paths in restricted hypercube-like networks.

The rest of this paper is organized as follows. In section 2, we introduce the definition of hypercube-like networks and present some of their properties. In section 3, we prove three results, generalizing the results in [22], [27] taking edge faults into consideration.

2. Motivation: survivable logical topology mapping in an IP-over-WDM optical network

As we pointed out in section 1, existence of disjoint path plays an important role in achieving fault tolerant communication in networks. In this section we briefly highlight one such application in the context of survivable logical topology mapping in an IP-over-WDM optical network.

The concept of layering plays an important role in the design of communication networks and protocols. An IP (Internet Protocol) over WDM (Wavelength Division Multiplexing) network is an example of a layered network. Here the WDM optical network is the physical layer represented by a graph G_P. The IP layer is the logical layer represented by a graph G_L. Without

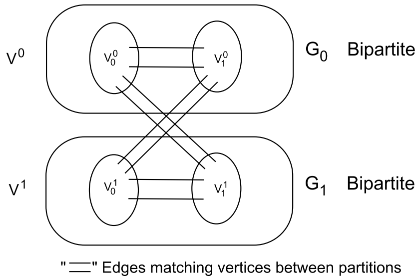

Fig. 1. Definition of B'_n.

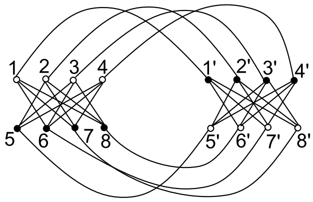

Fig. 2. An example in B'_4.

loss of generality we assume that G_I has the same vertex set as G_P. Also we assume that G_I and G_P are both 2-edge connected.

Each edge in G_I between vertices v and w corresponds to a path (called lightpath) between v and w in G_P [3]. To transmit information from vertex u to vertex v, first a u-v path P in G_I is identified. Then the information is transmitted using lightpaths corresponding to the logical links in P. If an edge in G_P fails, then several edges in G_I could fail causing G_I to become disconnected and thereby disrupting transmission of information. Survivable logical topology mapping (SLTM) is to map each edge in G_I into a lightpath in G_P such that a single edge failure in G_P does not disconnect G_I.

The SLTM problem has been studied using two approaches. The approach using mathematical programming formulation was pioneered in [25]. The other approach, called the structural approach, uses graph-theoretic concepts and was pioneered in [16].

The structural approach (though described differently in [16]) can be explained using the concept of ear decomposition of G_I. This approach may be viewed as constructing an ear decomposition of G_I and mapping the edges in each ear into edge-disjoint lightpaths in G_P. If no ear decomposition that admits such a mapping is available then the given SLTM problem is infeasible. Note that ears with a single edge are ignored. For details of this approach see [16,31] and [32].

There are several applications that require the existence of disjoint paths in networks.

3. Hypercube-like networks

Let $G_0 = (V_0, E_0), G_1 = (V_1, E_1)$ be two disjoint graphs with $|V_0| = |V_1|$. In addition, let ϕ be a bijection between V_0 and V_1 and $M = \{(v, \phi(v)) \mid v \in V_0\}$. We use $G_0 \oplus_M G_1$ to denote the graph $G = (V_0 \cup V_1, E_0 \cup E_1 \cup M)$.

The set of n-dimensional hypercube-like networks, denoted by HL_n, is a set of n-regular n-connected graphs with 2^n vertices and $n2^{n-1}$ edges that are defined recursively as follows.

1. $HL_1 = \{K_2\}$.
2. For $G_0, G_1 \in HL_n$, $G = G_0 \oplus_M G_1$ is a graph in HL_{n+1}.

We use $\bar{u}$ to denote the vertex in $V(G_{1-i})$ matched under ϕ. So $u = \bar{v}$ if $\bar{u} = v$. We next define the set of bipartite n-dimension hypercube-like networks, B'_n as follows:

(1) B'_1 is a complete graph with two vertices.
(2) For $i = 0, 1$, let G_i be a graph in B'_n with bipartition V_0^i and V_1^i. Let ϕ be a bijection between $V_0^0 \cup V_1^0$ and $V_0^1 \cup V_1^1$ such that $\phi(v) \in V_{1-i}^1$ if $v \in V_i^0$. Then $G = G_0 \oplus G_1$ is a graph in B'_{n+1}.

See Fig. 1 a pictorial description of B'_n. Also as an example B'_4 is shown in Fig. 2.

In Fig. 1, $V^0 = V_0^0 \cup V_1^0$ and $V^1 = V_0^1 \cup V_1^1$.

We next define the hypercube Q_n of dimension n. Q_n is a bipartite network whose vertex set is the set of 2^n binary strings of length n. Two vertices in Q_n are adjacent if and only if their labels differ in exactly one position. As an example, Q_4 is shown in Fig. 3. In this figure the binary strings representing i and i' differ in exactly one position.

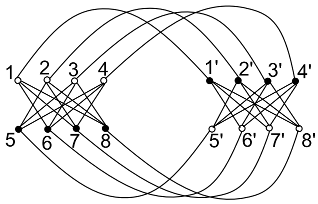

Fig. 3. Definition of Q_4.

Clearly $Q_n \in B'_n$. However $Q_n \neq B'_n$. For example, B'_4 in Fig. 2 and Q_4 in Fig. 3 are not isomorphic because only two cycles with length four containing edge $(8, 6')$ are in Fig. 2 but there exists three cycles of length four containing each edge in Q_4.

For a discussion of several properties of hypercubes and their variants see [4], [9] and [30].

We next present certain known results of B'_n that will be used in the next of the paper.

Theorem 1 *([27]). Every graph in B'_n is hamiltonian laceable and hamiltonian if $n \geq 2$.*

Theorem 2 *([27]). Let $n \geq 2$. Suppose that G is a graph in B'_n with bipartition V_0 and V_1. Let u_1 and u_2 be two distinct vertices in V_i and v_1 and v_2 are two distinct vertices in V_{1-i} with $i \in \{0, 1\}$. Then there exists a path P_1 joining u_1 to v_1 and a path P_2 joining u_2 to v_2 such that $V(P_1) \cup V(P_2) = V(G)$.*

Theorem 3 *([22]). Let G be a graph in B'_n with bipartition V_0 and V_1 for $n \geq 2$. Let z be vertex in V_i and u and v be two distinct vertices in V_{1-i} with $i \in \{0, 1\}$. Then there is a hamiltonian path of $G - \{z\}$ connecting u and v.*

Theorem 4 *([22]). Let n and k be any two positive integer with $k \leq n$. Let G be a graph in B'_n with bipartition V_0 and V_1. There exists a spanning (u, S)-fan in G for any vertex u in V_i and any vertex subset S with $|S| \leq n$ such that $|S \cap V_{1-i}| = 1$ with $i \in \{0, 1\}$.*

4. Spanning laceability of hypercube-like networks

In this section, we first prove two theorems which lead to our main result in Theorem 7. Theorem 1 in [27] is generalized in the following theorem.

Theorem 5. *Let G be a graph in B'_n with bipartition V_0 and V_1 for $n \geq 2$. Then G is $(n-2)$-edge fault tolerant hamiltonian laceable.*

Proof. Let $G = G_0 \oplus G_1$ in B'_n with V_0^i and V_1^i as the bipartition of G_i for every $i = 0, 1$. Let u and v be two arbitrary vertices where $u \in V_0^0 \cup V_0^1$ and $v \in V_1^0 \cup V_1^1$. Let F be an edge set of G with $|F| \leq n-2$.

For $n = 2$, the result holds trivially.

Suppose the result hold for $2 \leq k < n$.

For $n \geq 3$, we will prove the theorem by constructing a hamiltonian path of $G - F$ joining u to v. Let $F_i = F \cap E(G_i)$ and $f_i = |F_i|$ for $i = 0, 1$. We have the following cases.

Case 1. $\{u, v\}$ is in $V(G_0)$ or $\{u, v\}$ is in $V(G_1)$.

Assume that $u \in V_0^0$ and $v \in V_1^0$. This involves no loss of generality.

If $f_0 \leq n-3$ and $f_1 \leq n-3$, then by induction hypothesis, there exists a hamiltonian path P_0 joining u and v in $G_0 - F_0$ exists. We choose an edge $e = (x, y) \in V(P_0)$ such that $(x, \bar{x}) \notin F$ and $(y, \bar{y}) \notin F$; this is possible since $2^{n-2} > n-2 \geq |F|$ for $n \geq 3$. Let us rewrite P_0 as $P_0 = \langle u, P_{01}, x, y, P_{02}, v\rangle$. Induction implies that there exists a hamiltonian path P_1 joining $\bar{x}$ and $\bar{y}$ in $G_1 - F_1$. Then the path $P = \langle u, P_{01}, x, \bar{x}, P_1, \bar{y}, y, P_{02}, v\rangle$ is a required path in $G - F$. See Fig. 4.

If $f_0 = n-2$, then choose an edge $e = (x, y) \in F_0(= F)$. Let $F' = F \backslash (x, y)$. Induction implied existence of a hamiltonian path P_0 joining u and v in $G_0 - F'$. Let $e' = e$ if $e \in P_0$. Otherwise, arbitrarily select an edge e' in P_0. Without loss of generality, we assume that $e' = (x, y)$. Let us rewrite P_0 as $P_0 = \langle u, P_{01}, x, y, P_{02}, v\rangle$. Induction implies existence of a hamiltonian path P_1 joining $\bar{x}$ and $\bar{y}$ in G_1. Then the path $P = \langle u, P_{01}, x, \bar{x}, P_1, \bar{y}, y, P_{02}, v\rangle$ is a path in $G - F$. See Fig. 4.

If $f_1 = n-2$, then select an edge $e = (x, y) \in F_1(= F)$ with $|\{\bar{x}, \bar{y}\} \cap \{u, v\}| \leq 1$. Let $F' = F \backslash \{(x, y)\}$. Induction implies existence of a hamiltonian path P_1 joining x and y in $G_1 - F'$. If $\{\bar{x}, \bar{y}\} \cap \{u, v\} = \emptyset$, then, by Theorem 2, there exists a path P_{01} joining u and $\bar{x}$ and a path P_{02} joining v to $\bar{y}$ where $V(P_{01}) \cup V(P_{02}) = V(G_0)$. Then the path $P = \langle u, P_{01}, \bar{x}, x, P_1, y, \bar{y}, P_{02}, v\rangle$ is the desired path in $G - F$. See Fig. 5. If $\{\bar{x}, \bar{y}\} \cap \{u, v\} \neq \emptyset$, then assume that $u = \bar{y}$. By Theorem 3, there exists a path P_0 joining $\bar{x}$ and v in $G_0 - \{u\}$. Then the path $P = \langle u, y, P_1, x, \bar{x}, P_0, v\rangle$ is the desired path in $G - F$. See Fig. 6 for illustration.

Case 2. u is in $V(G_0)$ and v is in $V(G_1)$ or u is in $V(G_1)$ and v is in $V(G_0)$.

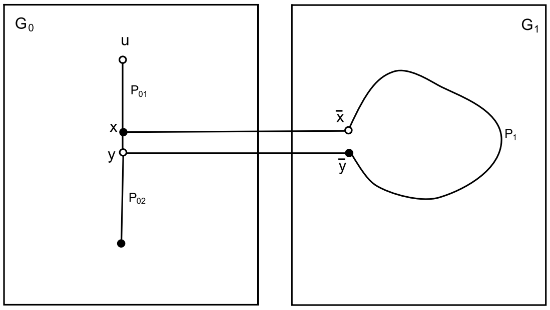

Fig. 4. Scenario for Case 1 in Theorem 5.

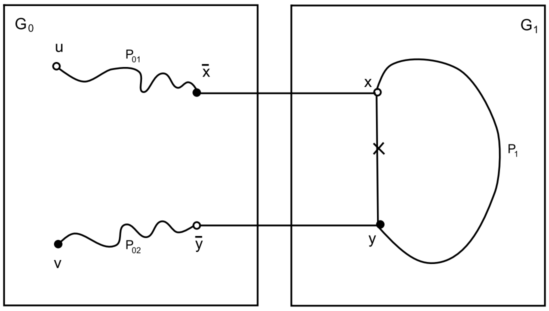

Fig. 5. Scenario for Case 1 in Theorem 5.

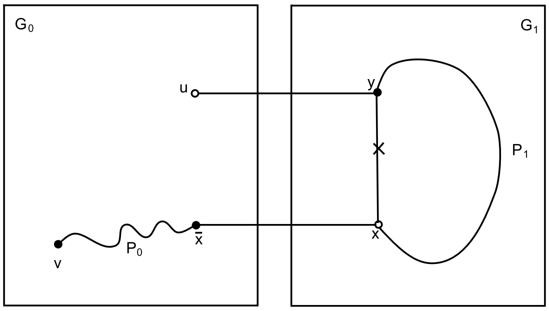

Fig. 6. Scenario for Case 1 in Theorem 5.

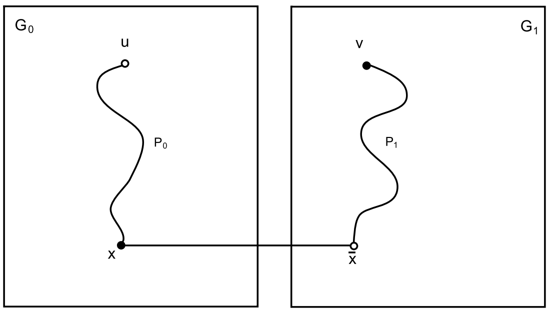

Fig. 7. Scenario for Case 2 in Theorem 5.

Without loss of generality let $u \in V_0^0$ and $v \in V_1^1$.

If $f_0 \le n-3$ and $f_1 \le n-3$, then select a vertex $x \in V_1^0$ with $(x, \bar{x}) \notin F$; this is possible since $2^{n-2} > n-2 \ge |F|$ for $n \ge 3$. Induction implies existence of a hamiltonian path P_0 joining u and x in $G_0 - F_0$ and a hamiltonian path P_1 joining v and $\bar{x}$ in $G_1 - F_1$. Then the path $P = \langle u, P_0, x, \bar{x}, P_1, u \rangle$ is a required path in $G - F$. See Fig. 7 for illustration.

Otherwise, $f_0 = n-2$ or $f_1 = n-2$. Assume then that $f_0 = n-2$. If there is an edge $e = (x, y) \in F_0 (= F)$ to which $\bar{v}$ is not adjacent, then choose a vertex $z \in V_1^0$ such that z is not adjacent with the edge (x, y) in F; this is possible since $2^{n-2} >$

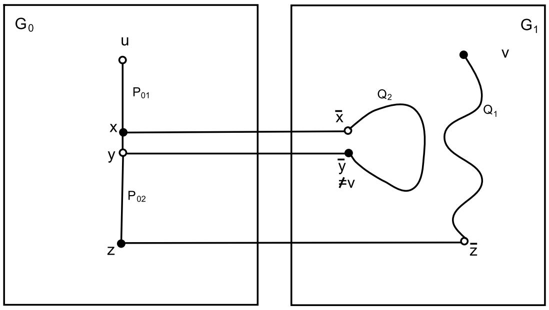

Fig. 8. Scenario for Case 2 in Theorem 5.

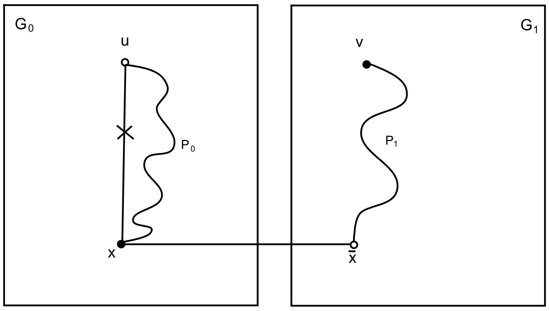

Fig. 9. Scenario for Case 2 in Theorem 5.

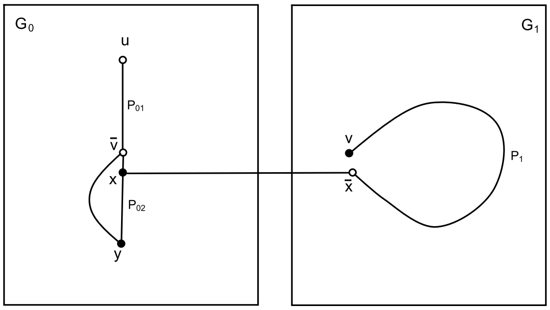

Fig. 10. Scenario for Case 2 in Theorem 5.

$n-2 \geq |F|$ for $n \geq 3$. Obviously, $(z, \bar{z}) \notin F$ for $f_0 = n-2$. Let $F' = F \backslash (x, y)$. Induction implies existence of a hamiltonian path P_0 joining u and z in $G_0 - F'$. Let $e' = e$ if $e \in E(P_0)$. Otherwise, select an edge $e' \in E(P_0)$ to which $\bar{v}$ is not adjacent. We also denote $e' = (x, y)$. Let us rewrite $P = \langle u, P_{01}, x, y, P_{02}, z\rangle$. By Theorem 2, there exists in G_1 a path Q_1 joining v to $\bar{z}$ and a path Q_2 joining $\bar{x}$ and $\bar{y}$ such that $V(Q_1) \cup V(Q_2) = V(G_1)$. Then the path $P' = \langle u, P_{01}, x, \bar{x}, Q_2, \bar{y}, y, P_{02}, z, \bar{z}, Q_1, v\rangle$ is a required path in $G - F$. See Fig. 8.

If $\bar{v}$ adjacent to all the edge in F, then let us select an edge $e = (\bar{v}, x) \in F$ and let $F' = F \backslash \{e\}$. If $\bar{v} = u$, then induction implies existence of a hamiltonian path P_0 in $G_0 - F'$ joining u and x and a hamiltonian path joining $\bar{x}$ and v in G_1. Then the path $\langle u, P_0, x, \bar{x}, P_1, v\rangle$ is the desired path in $G - F$. See Fig. 9. If $\bar{v} \neq u$, then we can choose a vertex $y \in N_{G_0}(\bar{v}) \backslash \{x\}$; this is possible since $|N_{G_0}(x)| = n-1 > |F|$. Induction implies existence of a hamiltonian path P_0 joining u and y in $G_0 - F'$. If $e \notin E(P)$, then the path $\langle u, P_0, y, \bar{y}, P_1, v\rangle$ is the desired path in $G - F$ where P_1 is a hamiltonian path joining $\bar{y}$ and v in G_1. Otherwise, let us rewrite $P_0 = \langle u, P_{01}, \bar{v}, x, P_{02}, y\rangle$ (Fig. 10) or $\langle u, P_{01}, x, \bar{v}, z, P_{02}, y\rangle$ (Fig. 11). If $P_0 = \langle u, P_{01}, \bar{v}, x, P_{02}, y\rangle$, then the path $\langle u, P_{01}, \bar{v}, y, P_{02}, x, \bar{x}, P_1, v\rangle$ is the desired path in $G - F$ where P_1 is the hamiltonian path joining $\bar{x}$ and v in G_1. See Fig. 10. If $\langle u, P_{01}, x, \bar{v}, z, P_{02}, y\rangle$, then $\langle u, P_{01}, x, \bar{x}, Q, \bar{z}, z, P_{02}, y, \bar{v}, v\rangle$ is a required path in $G - F$ where Q is the hamiltonian path joining $\bar{x}$ and $\bar{z}$ in $G_1 - \{v\}$ by Theorem 3. See Fig. 11.

This completes the proof of the theorem in all cases. □

As an example, Fig. 12 shows a Hamiltonian path indicated by dark edges between u and v with two faulty edges in B'_4.

We next present a generalization of Theorem 5 in [22].

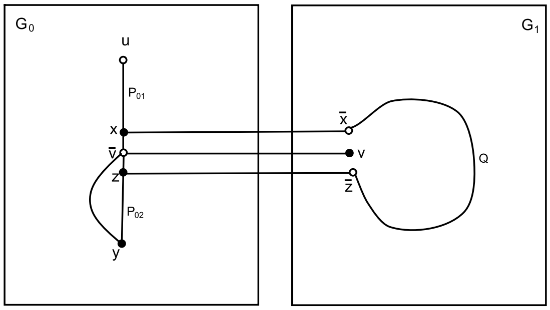

Fig. 11. Scenario for Case 2 in Theorem 5.

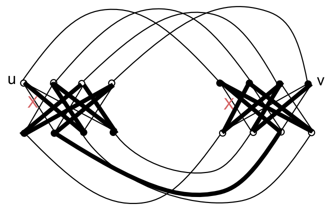

Fig. 12. Example for Theorem 5.

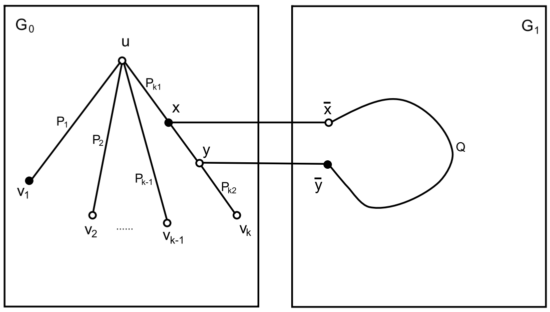

Fig. 13. Scenario for Case 1 in Theorem 6.

Theorem 6. *Let G be a graph in B'_n with bipartition V_0 and V_1 and F be an arbitrary edge set of G with $|F| \le n-2$. Then, in $G-F$, there exists a spanning (u,S)-fan in $G-F$ for any $u \in V_i$ and $S \subseteq V(G) \setminus \{u\}$ with $|S| \le n-|F|$ and $|S \cap V_{1-i}| = 1$ with $i \in \{0,1\}$.*

Proof. Let $G = G_0 \oplus G_1$ in B'_n with V_0^i and V_1^i be the bipartition of G_i for $i = 0, 1$. Note that $V(G_i) = V_0^i \cup V_1^i$. See Fig. 4. We know that G is a n-regular, n-connected non-complete graph. Let $F \subseteq E(G)$ with $|F| \le n-2$. Then pick any vertex u in V_0^0 and let $S = \{v_1, v_2, \cdots, v_k\}$ be any vertex subset in $V(G) \setminus \{u\}$ with v_1 being the unique vertex in $(V_1^0 \cup V_1^1) \cap S$ where $k \le n - |F|$. Let $S_i = S \cap V(G_i)$, $F_i = F \cap E(G_i)$ for $i = 0, 1$ and $F_2 = F \setminus (F_0 \cup F_1)$. To prove the result, we need to show a spanning (u,S)-fan in $G-F$.

We prove the statement by induction. By Theorem 4 and Theorem 5, the result holds for $n = 3$. We assume that $n \ge 4$. If $|F| = 0$, then the result holds by Theorem 4. If $|F| = n-2$, then $k = |S| \le 2$. By Theorem 5, we get the result. So, we only consider the case $1 \le |F| \le n-3$ and prove the theorem according to the following cases.

Case 1. $|S_1| = 0$. In this case $S_0 = S$ and $v \in V_1^0$.

If F is a proper subset of $E(G_0)$, then arbitrarily choose an edge $e \in F$ and let $F' = F \setminus \{e\}$. Then $0 \le |F'| \le n-4$ and $|F'| + |S_0| \le n-1$. Induction implies existence of a spanning (u,S)-fan $\{P_1, P_2, \cdots, P_k\}$ in $G_0 - F'$. Let $e' = e$ if $e \in \bigcup_{i=1}^k E(P_i)$. Otherwise, arbitrarily choose an edge $e' \in \bigcup_{i=1}^k E(P_i)$. Without loss of generality, we assume that $e' = (x, y) \in E(P_k)$ and denote $P_k = \langle u, P_{k1}, x, y, P_{k2}, v_k \rangle$. Since $|F_1| \le |F| \le n-3$, by induction statement, there exists a Hamiltonian path Q between $\bar{x}$ and $\bar{y}$ in G_1. We set $R_i = P_i$ for $1 \le i \le k-1$, $R_k = (u, P_{k1}, x, \bar{x}, Q, \bar{y}, y, P_{k2}, v_k)$. Then $\{R_1, R_2, \ldots, R_k\}$ forms a required spanning fan in $G-F$. See Fig. 13.

If $F \not\subseteq E(G_0)$, then $0 \le |F_0| \le n-4$ and $|F_0| + |S_0| \le n-1$. Induction implies existence of a spanning (u,S)-fan $\{P_1, P_2, \cdots, P_k\}$ in $G_0 - F_0$. We can choose an edge $e = (x, y) \in \bigcup_{i=1}^k E(P_i)$ such that $(x, \ \bar{x}) \notin F$ and $(y, \ \bar{y}) \notin F$; this is guaranteed since $2^{n-2} > n-3 \ge |F|$ for $n \ge 4$. Then assume that $e \in E(P_k)$ and denote $P_k = \langle u, P_{k1}, x, y, P_{k2}, v_k \rangle$. Since

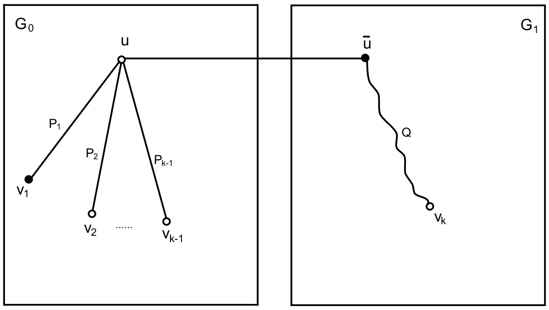

Fig. 14. Scenario for Subcase 2.1 in Theorem 6.

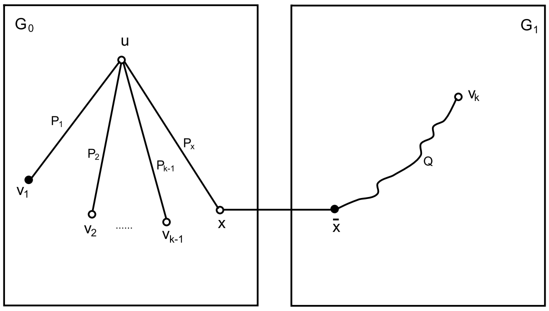

Fig. 15. Scenario for Subcase 2.1 in Theorem 6.

$|F_1| \le |F| \le n-3$, by induction, there exists a Hamiltonian path Q between $\bar{x}$ and $\bar{y}$ in $G_1 - F_1$. We set $R_i = P_i$ for $1 \le i \le k-1$, $R_k = (u, P_{k1}, x, \bar{x}, Q, \bar{y}, y, P_{k2}, v_k)$. Then $\{R_1, R_2, \ldots, R_k\}$ forms a required spanning fan in $G - F$. See Fig. 13.

Case 2. $|S_1| = 1$.

In this case, we have $|S_0| = k-1$, $|F_0| + |S_0| \le n-1$ and $|F_1| + |S_1| \le |F| + 1 \le n-2$.

Subcase 2.1. v_1 is not in S_1.

In this subcase, we assume that $S_1 = \{v_k\}$. If $(u, \bar{u}) \notin F$, then induction implies existence of a spanning (u, S_0)-fan $\{P_1, P_2, \cdots, P_{k-1}\}$ in $G_0 - F_0$ and a Hamiltonian path Q between $\bar{u}$ and v_k in $G_1 - F_1$. We set $R_i = P_i$ for $1 \le i \le k-1$, $R_k = (u, \bar{u}, Q, v_k)$. Then $\{R_1, R_2, \ldots, R_k\}$ forms a required spanning fan in $G - F$. See Fig. 14. If $(u, \bar{u}) \in F$, then $|F_0| + |S_0| \le n-2$. We can choose an edge $(x, \bar{x}) \notin F$ with $x \in V_0^0$ for $2^{n-2} > n-3 \ge |F|$ when $n \ge 4$. Induction implies existence of a spanning $(u, S_0 \cup \{x\})$-fan $\{P_1, P_2, \cdots, P_{k-1}, P_x\}$ in $G_0 - F_0$ and a Hamiltonian path Q between $\bar{x}$ and v_k in $G_1 - F_1$. We set $R_i = P_i$ for $1 \le i \le k-1$, $R_k = (u, P_x, x, \bar{x}, Q, v_k)$ Then $\{R_1, R_2, \ldots, R_k\}$ forms a required spanning fan in $G - F$. See Fig. 15.

Subcase 2.2. $v_1 \in S_1$. So $v_1 \in V_1^1$.

In this subcase, let $S_0 = \{v_2, v_3, \cdots, v_k\}$ and $S_1 = \{v_1\}$.

If $F_0 \subset F$, then $|F_0| + |S_0| \le n-2$. Let us choose an edge $(x, \bar{x}) \notin F$ such that $x \in V_1^0$ for $2^{n-2} > n-3 \ge |F|$ when $n \ge 4$. Induction implies existence of a spanning $(u, \{x\} \cup S_0)$-fan $\{P_x, P_2, \cdots, P_{k-1}, P_k\}$ in $G_0 - F_0$ and a Hamiltonian path Q between $\bar{x}$ and v_1 in $G_1 - F_1$. We set $R_i = P_i$ for $2 \le i \le k$, $R_1 = (u, P_x, x, \bar{x}, Q, v_1)$. Then $\{R_1, R_2, \ldots, R_k\}$ forms a required spanning fan in $G - F$. See Fig. 16.

If $F_0 = F$ and $\bar{v}_1$ is not adjacent to all the fault edges in F_0, then choose an edge $e = (x, y) \in F_0$ and $z \in V_1^0$ with $x \in V_1^0$ and $v_1 \notin \{\bar{y}, \bar{z}\}$. Let $F_0' = F_0 \setminus \{e\}$. Then $|F_0'| = |F| - 1 \le n-4$ and $(|S_0| + 1) + |F_0'| \le n-1$. Induction implies existence of a spanning $(u, \{z\} \cup S_0)$-fan $\{P_z, P_2, \cdots, P_{k-1}, P_k\}$ in $G_0 - F_0'$. Let $e' = e$ if $e \in \bigcup_{i=2}^{k} E(P_i) \cup E(P_z)$. Otherwise, choose an edge $e' \in \bigcup_{i=2}^{k} E(P_i) \cup E(P_z)$ to which $\bar{v}_1$ is not adjacent and denote $e' = (x, y)$. By Theorem 2, there exist two disjoint paths Q_1 and Q_2 of G_1 such that (1) Q_1 joins $\bar{x}$ and $\bar{y}$, (2) Q_2 joins $\bar{z}$ and v_1, and (3) $V(Q_1) \cup V(Q_2) = V(G_1)$. If $e' \in E(P_z)$ and $P_z = \langle u, P_{z1}, x, y, P_{z2}, z\rangle$, we set $R_i = P_i$ for $2 \le i \le k$, $R_1 = (u, P_{z1}, x, \bar{x}, Q_1, \bar{y}, y, P_{z2}, z, \bar{z}, Q_2, v_1)$. Then $\{R_1, R_2, \ldots, R_k\}$ forms a required spanning fan in $G - F$. See Fig. 17. Otherwise let $e' \in E(P_k)$ and $P_k = \langle u, P_{k1}, x, y, P_{k2}, v_k\rangle$. We set $R_i = P_i$ for $2 \le i \le k-1$, $R_1 = (u, P_z, z, \bar{z}, Q_2, v_1)$ and $R_k = (u, P_{k1}, x, \bar{x}, Q_1, \bar{y}, y, P_{k2}, v_k)$. Then $\{R_1, R_2, \ldots, R_k\}$ forms a required spanning fan in $G - F$. See Fig. 18.

If $F_0 = F$ and $\bar{v}_1$ is adjacent to all fault edges, then arbitrarily choose an edge $e = (x, \bar{v}_1) \in F_0$ such that $x \in V_1^0$. Let $F_0' = F_0 \setminus \{e\}$. Then $|F_0'| = |F| - 1 \le n-4$ and $(|S_0| + 1) + |F_0'| \le n-1$. Induction implies existence of a spanning $(u, \{x\} \cup S_0)$-fan $\{P_x, P_2, \cdots, P_{k-1}, P_k\}$ in $G_0 - F_0'$. If $e \notin E(P_x)$, then we set $R_i = P_i$ for $2 \le i \le k$, $R_1 = (u, P_x, x, \bar{x}, Q, v_1)$ where Q is a Hamiltonian path between $\bar{x}$ and v_1 in G_1. See Fig. 16. Otherwise, $e \in E(P_x)$. Since G_0 is $(n-1)$-regular and $|F| \le$

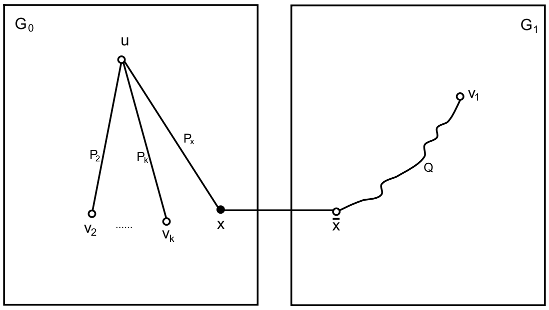

Fig. 16. Scenario for Subcase 2.2 in Theorem 6.

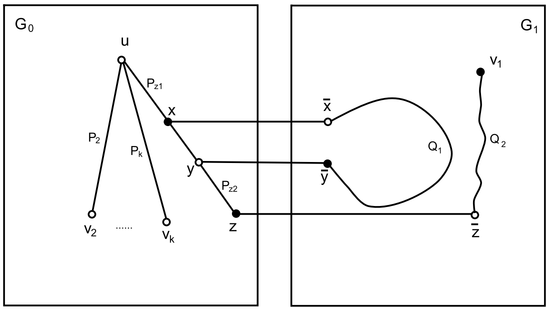

Fig. 17. Scenario for Subcase 2.2 in Theorem 6.

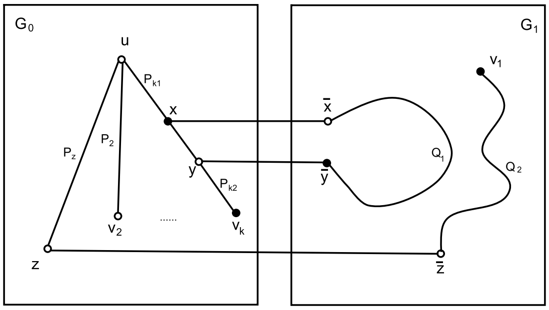

Fig. 18. Scenario for Subcase 2.2 in Theorem 6.

$n-3$, then x has a neighbor y with $(x, y) \notin F$. If $y \in V(P_1)$, then we let $P_1 = \langle u, P_{x1}, y, z, P_{x2}, \bar{v}_1, x\rangle$. We set $R_i = P_i$ for $2 \le i \le k$, $R_1 = (u, P_{x1}, y, x, \bar{x}, Q, \bar{z}, z, P_{x2}, \bar{v}_1, v_1)$ where Q is a Hamiltonian path between $\bar{x}$ and $\bar{z}$ in $G_1 - v_1$ by Theorem 2. See Fig. 19. Otherwise, assume $y \in V(P_k)$. Let $P_x = \langle u, P_{x1}, \bar{v}_1, x\rangle$. Let $P_k = \langle u, P_{k1}, z, v_k\rangle$ if $y = v_k$ (Fig. 20) or $P_k = \langle u, P_{k1}, y, z, P_{k2}, v_k\rangle$ (Fig. 21). We set $R_i = P_i$ for $2 \le i \le k-1$, $R_1 = (u, P_{x1}, \bar{v}_1, v_1)$, $R_k = (u, P_{k1}, z, \bar{z}, Q, \bar{x}, x, v_k)$ or $R_k = (u, P_{k1}, y, x, \bar{x}, Q, \bar{z}, z, P_{k2}, v_k)$ where Q is a Hamiltonian path between $\bar{x}$ and $\bar{z}$ in $G_1 - \{v_1\}$ by Theorem 2. Then $\{R_1, R_2, \ldots, R_k\}$ forms a required spanning fan in $G - F$. See Fig. 20 and Fig. 21 for illustration. The case when $\bar{v}_1 \notin P_k$ can be treated similarly.

Case 3. $|S_1| \ge 2$.

Subcase 3.1. $v_1 \notin S_1$. So $v_1 \in V_1^0$.

In this subcase, we assume that $S_0 = \{v_1, v_2, \cdots, v_t\}$ and $S_1 = \{v_{t+1}, \cdots, v_k\}$. Then $|S_0| \le n-2-|F|$ and $|S_1| \le n-1-|F|$.

If $F_0 = F$ or $F_1 = F$, then $(u, \bar{u}) \notin F$. Let $A = \{(v_k, \bar{v}_i) \,|\, \text{if } (v_k, \bar{v}_i) \text{ is an edge for } v_i \in S_0 \setminus \{v_1\}\}$. Then $|F_1| + |A| \le |F_1| + (|S_0| - 1) \le n - |S_1| - 1 \le n-3$ and $|F_1| + |A| + |S_1| \le n-1$. Induction implies existence of a spanning $(v_k, \{\bar{u}\} \cup S_1 \setminus \{v_k\})$-fan $\{P_{t+1}, P_{t+2}, \cdots, P_{k-1}, P_k\}$ in $G_1 - F_1 - A$. Moreover, let $P_i = \langle v_k, x_i, P_{i1}, v_i\rangle$ for $t+1 \le i \le k-1$ and $P_k = \langle v_k, P_k, \bar{u}\rangle$. Let $X = \cup_{i=t+1}^{k-1} x_i$. Obviously, $X \cap S_0 = \emptyset$, $|S_0| + |X| \le |S| - 1$ and $|F| + (|S_0| + |X|) \le n-1$. Induction implies existence of a spanning $(u, S_0 \cup X)$-fan $\{Q_1, \cdots, Q_t, Q_{t+1}, \cdots, Q_{k-1}\}$ in $G_0 - F_0$. We set $R_i = Q_i$ for $1 \le i \le t$, $R_j = (u, Q_j, \bar{x}_j, x_j, P_j, v_j)$ for $t+1 \le j \le k-1$ and $R_k = (u, \bar{u}, P_k, v_k)$. Then $\{R_1, R_2, \ldots, R_k\}$ forms a required spanning fan in $G - F$. See Fig. 22 for illustration.

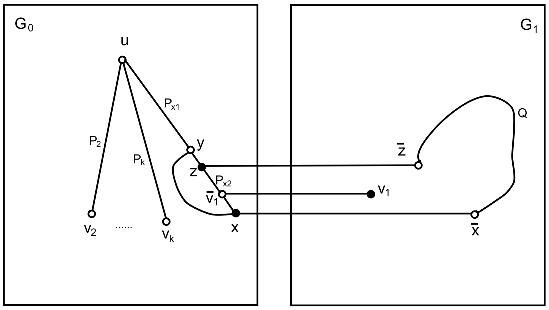

Fig. 19. Scenario for Subcase 2.2 in Theorem 6.

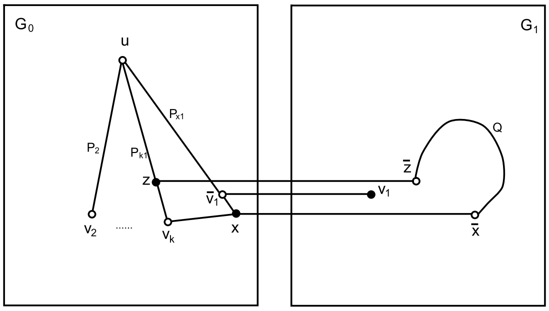

Fig. 20. Scenario for Subcase 2.2 in Theorem 6.

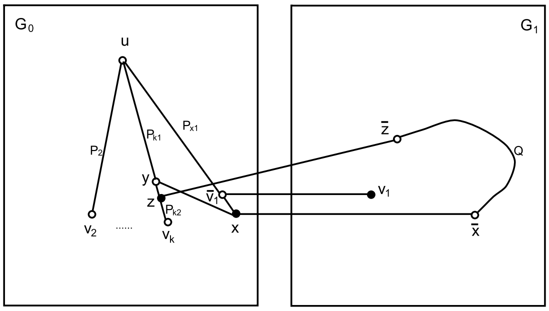

Fig. 21. Scenario for Subcase 2.2 in Theorem 6.

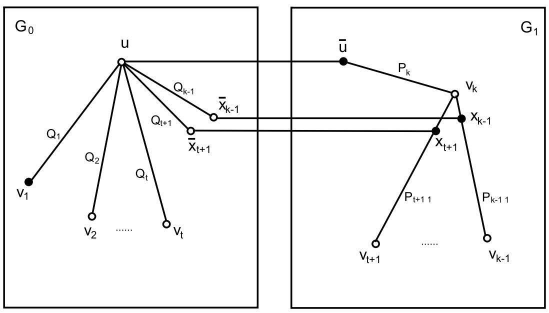

Fig. 22. Scenario for Subcase 3.1 in Theorem 6.

If $F_0 \subset F$ and $F_1 \subset F$, then choose $x \in V_0^0 \setminus S_0$ with $(x, \bar{x}) \notin F$. Let $A = \{(v_k, \bar{v}_i) \,|\, \text{if } (v_k, \bar{v}_i)$ be an edge for $v_i \in S_0 \setminus \{v_1\}\}$ and $B = \{(v_k, \bar{y}) \,|\, \text{if } (v_k, \bar{y})$ is an edge for $(y, \bar{y}) \in F_2\}$. Then $|A| \le |S_0| - 1$, $|B| \le |F_2|$, and $|A| + |B| + |F_1| \le (|S_0| - 1) + |F_2| + |F_1| \le n - 1 - |S_1| \le n - 3$. Induction implies existence of a spanning $(v_k, \{\bar{x}\} \cup S_1 \setminus \{v_k\})$-fan $\{P_{t+1}, P_{t+2}, \cdots, P_{k-1}, P_{\bar{x}}\}$ in $G_1 - F_1 - A - B$. Moreover, let us rewrite $P_i = \langle v_k, x_i, P_{i1}, v_i \rangle$ for $t+1 \le i \le k-1$ and $P_k = \langle v_k, P_{\bar{x}}, \bar{x} \rangle$. Let $X = \cup_{i=t+1}^{k-1} \bar{x}_i \cup \{x\}$. Obviously, $X \cap S_0 = \emptyset$, $|S_0| + |X| = |S|$ and $|F_0| + (|S_0| + |X|) \le n-1$. Induction implies existence of a spanning $(u, S_0 \cup X)$-fan

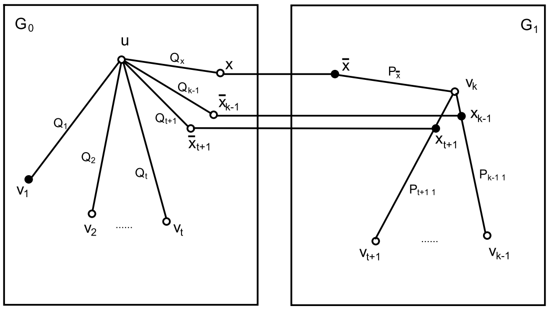

Fig. 23. Scenario for Subcase 3.1 in Theorem 6.

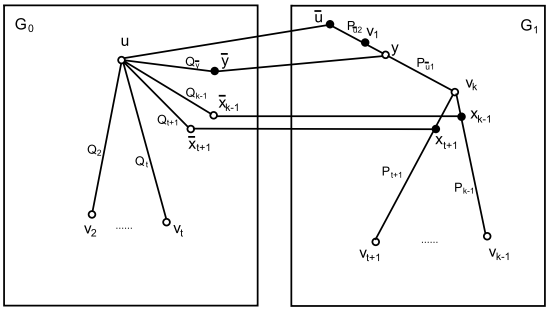

Fig. 24. Scenario for Subcase 3.2 in Theorem 6.

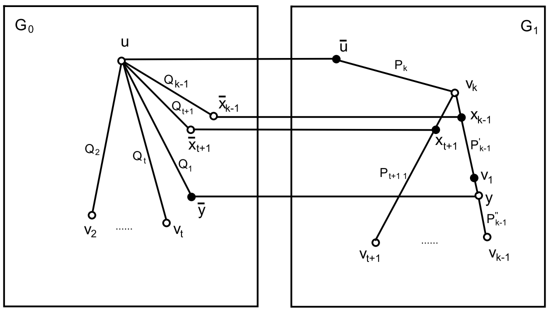

Fig. 25. Scenario for Subcase 3.2 in Theorem 6.

$\{Q_1, \cdots, Q_t, Q_{t+1}, \cdots, Q_{k-1}, Q_x\}$ in $G_0 - F_0$. We set $R_i = Q_i$ for $1 \le i \le t$, $R_j = (u, Q_j, \bar{x}_j, x_j, P_{j1}, v_j)$ for $t+1 \le j \le k-1$ and $R_k = (u, Q_x, x, \bar{x}, P_{\bar{x}}, v_k)$. Then $\{R_1, R_2, \ldots, R_k\}$ forms a desired spanning fan in $G - F$. See Fig. 23.

Subcase 3.2. $v_1 \in S_1$. So $v_1 \in V_1^1$

In this subcase, we assume that $S_0 = \{v_2, \cdots, v_t\}$ and $S_1 = \{v_1, v_{t+1}, \cdots, v_k\}$. Then $|S_0| \le n - 2 - |F| \le n - 3$ and $|S_1| \le n - |F| \le n - 1$.

If $F_0 = F$, then $F_1 \cup F_2 = \emptyset$. Let $A = \{(v_k, \bar{v}_i) \mid \text{if } (v_k, \bar{v}_i) \text{ be an edge for } v_i \in S_0\}$. Then $|A| \le |S_0| \le |S| - 2 \le n - |F| - 2 \le n - 3$ and $|A| + |S_1| \le |S_0| + |S_1| \le n - |F| \le n - 1$. Induction implies existence of a spanning $(v_k, \{\bar{u}\} \cup S_1 \setminus \{v_k\})$-fan $\{P_{t+1}, P_{t+2}, \cdots, P_{k-1}, P_{\bar{u}}\}$ in $G_1 - A$. If $v_1 \in V(P_k)$, then let $P_i = \langle v_k, x_i, P_{i1}, v_i \rangle$ for $t+1 \le i \le k-1$ and $P_{\bar{u}} = \langle v_k, P_{\bar{u}1}, y, v_1, P_{\bar{u}2}, \bar{u} \rangle$. Let $X = \cup_{i=t+1}^{k-1} \bar{x}_i \cup \{\bar{y}\}$. Obviously, $X \cap S_0 = \emptyset$, $|S_0| + |X| = |S| - 1$ and $|F| + (|S_0| + |X|) \le n - 1$. Induction implies existence of a spanning $(u, S_0 \cup X)$-fan $\{Q_2, \cdots, Q_t, Q_{t+1}, \cdots, Q_{k-1}, Q_{\bar{y}}\}$ in $G_0 - F_0$. We set $R_i = P_i$ for $2 \le i \le t$, $R_j = (u, Q_j, \bar{x}_j, x_j, P_{j1}, v_j)$ for $t+1 \le j \le k-1$, $R_1 = (u, \bar{u}, P_{\bar{u}2}, v_1)$ and $R_k = (u, Q_{\bar{y}}, \bar{y}, y, P_{\bar{u}1}, v_k)$. Then $\{R_1, R_2, \ldots, R_k\}$ forms a desired spanning fan in $G - F$. See Fig. 24. If $v_1 \in \cup_{i=t+1}^{k-1} V(P_i)$, we assume that $v_1 \in V(P_{k-1})$. Let $P_{k-1} = \langle v_k, x_{k-1}, P'_{k-1}, v_1, y, P''_{k-1}, v_{k-1} \rangle$, $P_i = \langle v_k, x_i, P_{i1}, v_i \rangle$ for $t+1 \le i \le k-2$ and $P_k = \langle v_k, P_k, \bar{u} \rangle$. Let $X = \cup_{i=t+1}^{k-1} \bar{x}_i \cup \{\bar{y}\}$. Obviously, $X \cap S_0 = \emptyset$, $|S_0| + |X| = |S| - 1$ and $|F| + (|S_0| + |X|) \le n - 1$. Induction implies existence of a spanning $(u, S_0 \cup X)$-fan $\{Q_1, Q_2, \cdots, Q_t, Q_{t+1}, \cdots, Q_{k-1}\}$ in $G_0 - F_0$. We set $R_i = P_i$ for $2 \le i \le t$, $R_j = (u, Q_j, \bar{x}_j, x_j, P_{j1}, v_j)$ for $t+1 \le j \le k-2$, $R_1 = (u, Q_{k-1}, \bar{x}_{k-1}, x_{k-1}, P'_{k-1}, v_1)$, $R_{k-1} = (u, Q_1, \bar{y}, y, P''_{k-1}, v_{k-1})$ and $R_k = (u, \bar{u}, P_k, v_k)$. Then $\{R_1, R_2, \ldots, R_k\}$ forms a required spanning fan in $G - F$. See Fig. 25.

If $F_0 \subset F$, we will discuss this subcase according to $|S_2| = 2$ and $|S_2| \ge 3$.

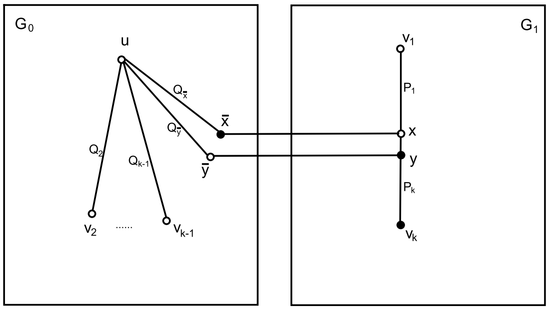

Fig. 26. Scenario for Subcase 3.2 in Theorem 6.

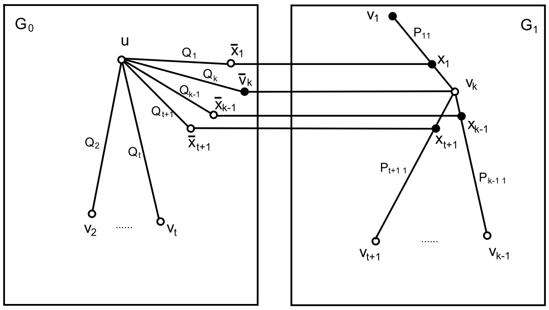

Fig. 27. Scenario for Subcase 3.2 in Theorem 6.

If $|S_1| = 2$, we assume that $S_1 = \{v_1, v_k\}$. Induction implies existence of a Hamiltonian path Q between v_1 and v_k in $G_1 - F_1$. Then we can select an edge $(x, y) \in E(P)$ with $(x, \bar{x}) \notin F_2$, $(y, \bar{y}) \notin F_2$ and $\bar{y} \notin S_0 \cup \{u\}$ for $2^{n-2} > n - 1 \geq |S_0| + |F| + 1$ for $n \geq 4$. Let $P = (v_1, P_1, x, y, P_k, v_k)$. Since $F_0 \subset F$, induction implies existence of a spanning $(u, \{\bar{x}, \bar{y}\} \cup S_0\})$-fan $\{Q_{\bar{x}}, Q_2, \cdots, Q_{k-1}, Q_{\bar{y}}\}$ in $G_0 - F_0$. We set $R_i = Q_i$ for $2 \leq i \leq k-1$, $R_1 = (u, Q_{\bar{x}}, \bar{x}, x, P_1, v_1)$ and $R_k = (u, Q_{\bar{y}}, \bar{y}, y, P_k, v_k)$. Then $\{R_1, R_2, \ldots, R_k\}$ forms a desired spanning fan in $G - F$. See Fig. 26.

If $|S_1| \geq 3$, let $S_1 = \{v_1, v_{t+1}, \cdots, v_k\}$. If $(v_k, \bar{v}_k) \notin F$, let $A = \{(v_k, \bar{v}_i) \,|\, \text{if } (v_k, \bar{v}_i) \text{ be an edge for } v_i \in S_0\}$ and $B = \{(v_k, \bar{y}) \,|\, \text{if } (v_k, \bar{y}) \text{ is an edge for } (y, \bar{y}) \in F_2\}$. Then $|A| \leq |S_0|$, $|B| \leq |F_2|$, and $|A| + |B| + |F_1| \leq n - |S_1| \leq n - 3$. Induction implies existence of a spanning $(v_k, S_1 \setminus \{v_k\})$-fan $\{P_1, P_{t+1}, P_{t+2}, \cdots, P_{k-1}\}$ in $G_1 - F_1 - A - B$. Moreover, let $P_1 = \langle v_k, x_1, P_{11}, v_1\rangle$ and $P_i = \langle v_k, x_i, P_{i1}, v_i\rangle$ for $t + 1 \leq i \leq k - 1$. Let $X = \cup_{i=t+1}^{k-1} \bar{x}_i \cup \{\bar{x}_1, \ \bar{v}_k\}$. Obviously, $X \cap S_0 = \emptyset$, $|S_0| + |X| = |S|$ and $|F_0| + (|S_0| + |X|) \leq n - 1$. Induction implies existence of a spanning $(u, S_0 \cup X)$-fan $\{Q_1, \cdots, Q_t, Q_{t+1}, \cdots, Q_k\}$ in $G_0 - F_0$. We set $R_i = Q_i$ for $2 \leq i \leq t$, $R_j = (u, Q_j, \bar{x}_j, x_j, P_{j1}, v_j)$ for $t + 1 \leq j \leq k - 1$, $R_1 = (u, Q_1, \bar{x}_1, x_1, P_{11}, v_1)$ and $R_k = (u, Q_k, \bar{v}_k, v_k)$. Then $\{R_1, R_2, \ldots, R_k\}$ forms a required spanning fan in $G - F$. See Fig. 27.

If $(v_k, \bar{v}_k) \in F$, then $|F_1| < |F| \leq n-3$. Choose $(x, \bar{x}) \notin F$ with $x \in V_0^1$. Let $A = \{(x, \bar{v}_i) \,|\, \text{if } (\bar{x}, \bar{v}_i) \text{ be an edge where } v_i \in S_0\}$ and $B = \{(v_k, \bar{y}) \,|\, \text{if } (v_k, \bar{y}) \text{ is an edge for } (y, \bar{y}) \in F_2 \setminus \{(v_k, \bar{v}_k)\}\}$. Then $|A| \leq |S_0|$, $|B| \leq |F_2| - 1$, and $|A| + |B| + |F_1| \leq n - 1 - |S_1| \leq n - 4$. Induction implies existence of a spanning (x, S_1)-fan $\{P_{t+1}, P_{t+2}, \cdots, P_k\}$ in $G_1 - F_1 - A - B$. Moreover, we write $P_1 = \langle x, P_1, v_1\rangle$ and $P_i = \langle x, x_i, P_{i1}, v_i\rangle$ for $t + 1 \leq i \leq k$. Let $X = \cup_{i=t+1}^{k} \bar{x}_i \cup \{\bar{x}\}$. Obviously, $X \cap S_0 = \emptyset$, $|S_0| + |X| = |S|$ and $|F_0| + (|S_0| + |X|) \leq n - 1$. Induction implies existence a spanning $(u, S_0 \cup X)$-fan $\{Q_{\bar{x}}, Q_2, \cdots, Q_t, Q_{t+1}, \cdots, Q_k\}$ in $G_0 - F_0$. We set $R_i = Q_i$ for $2 \leq i \leq t$, $R_j = (u, Q_j, \bar{x}_j, x_j, P_{j1}, v_j)$ for $t + 1 \leq j \leq k$ and $R_1 = (u, Q_{\bar{x}}, \bar{x}, x, P_1, v_1)$. Then $\{R_1, R_2, \ldots, R_k\}$ forms a required spanning fan in $G - F$. See Fig. 28.

We have shown that the result is true in all cases. □

As an example, Fig. 29 shows a spanning fan indicated by dark and dotted edges between u and $\{v_1, \ v_2\}$ with two faulty edges in B'_4.

The following theorem generalizes Theorem 6 in [22].

Theorem 7. *Every graph in B'_n is f-edge fault-tolerant w^*-laceable for every $0 \leq f \leq n - 2$ and $1 \leq w \leq n - f$.*

Proof. Let $G = G_0 \oplus G_1$ in B'_n with V_0^i and V_1^i be the bipartition of G_i for every $i = 0, 1$. Let F be an edge set such that $f = |F| \leq n - 2$. Let u be any vertex in V_0 and v be any vertex in V_1. We will show that there exists a w^*-container of G between u and v in $G - F$ for every $w \leq n - f$. For $w \leq n - f$, we can choose $\{x_1, x_2, \cdots, x_{w-1}\} \subseteq N_G(v)$ such that $(v, x_i) \notin F$. Let $S = \{v, x_1, \cdots, x_{w-1}\}$. Thus, there exists a spanning (u, S)-fan $\{Q_1, Q_2, \cdots, Q_w\}$ in $G - F$ by Theorem 6. We set $P_1 = Q_1$, $P_i = \langle u, Q_i, x_i, v\rangle$. Then $\{P_1, P_2, \cdots, P_w\}$ is the w^*-container of $G - F$ between u and v. □

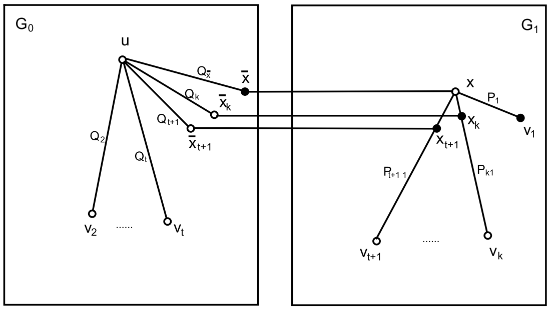

Fig. 28. Scenario for Subcase 3.2 in Theorem 6.

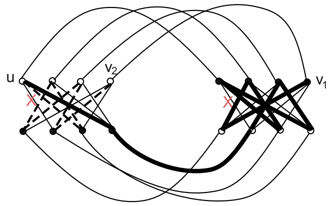

Fig. 29. Example for Theorem 6.

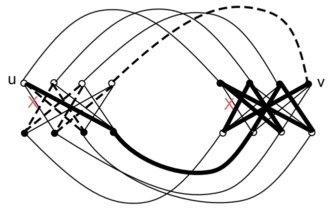

Fig. 30. Example for Theorem 7.

As an example, Fig. 30 shows a 2*-container indicated by dark and dotted edges between u and v with two faulty edges in B'_4.

Remark. Let G be an n-regular graph and u be a vertex in G. Choose a vertex $v \in N_G(u)$. Let F be an edge set of G such that all edges in F are adjacent with u and $|F| = n - 1$. Thus, there is no hamiltonian path between u and v in $G - F$. It shows that our result in Theorem 7 is optimal.

5. Summary

In this paper, we have studied fault tolerance properties of hypercube-like networks and established three spanning laceability properties. These results are in Theorem 5, 6 and 7. Our results generalize those properties of hypercube-like networks without edge faults presented in [22], [27]. Since hypercube is a special class of hypercube-like networks, these fault tolerance properties hold for hypercube as well.

As study in [4] and [9], [26], [28], [29] draws our future research attention the existence of internally disjoint paths between specified pairs of vertices. An interesting problem for future research is to study such problems for hypercube-like networks.

As to another problem for future research, let us return to the survivable logical topology routing problem as described in section 2. In this problem we are given a physical topology (an optical network) G_P and a logical topology (IP layer) G_I. Assume that G_P and G_I have the same vertex set. Logical topology routing requires each logical link (x, y) to be mapped into a path between the vertices x and y in G_P. When a physical link fails, several logical links could fail. So the problem is to find a mapping of the logical links so that a physical link failure does not cause G_I to be disconnected. It is desirable to achieve a mapping that protects G_I against multiple physical link failures. This problem is NP-complete. An interesting question is to design efficient heuristics to achieve such a mapping. We believe that this is possible if a hypercube-like network is used as a logical topology because of the fault tolerant laceability properties established in this paper.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper. The authors declared that they have no conflicts of interest to this work.

We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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