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Conditional diagnosability of a class of matching composition networks under the comparison model [☆,☆☆]

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ABSTRACT

Fault diagnosis of interconnection networks is an important consideration in the design and maintenance of multiprocessor systems. Herein, we study fault diagnosis, which is the identification of faulty processors in high speed parallel processing systems. Conditional diagnosability, proposed by Lai et al. [22], assumes that no fault set can contain all the neighbors of any processor in a system; this is a well-accepted and general measure of the diagnosis ability of an interconnection network of multiprocessor systems. The diagnosability and conditional diagnosability of many interconnection networks have been studied using various diagnosis models. In this paper we study the conditional diagnosability of matching composition networks under the comparison model (MM^* model). In [31] Yang determined a set of sufficient conditions for a network G to be conditionally $(3n-3-C(G))$-diagnosable. Our main contribution in this paper is to extend Yang's result by determining a larger class of networks that are conditionally $(3n-3-C(G))$-diagnosable. Yang's result [31] and earlier results for the hypercube, the crossed cube, the twisted cube and the Möbius cube [18,32,33] all become corollaries of our main result. Thus this paper extends the state of the art in the area of conditional diagnosability of multiprocessor systems.

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1. Introduction

Continuous advances in semiconductor technology have made it possible to develop very large digital systems comprising of hundreds of thousands of components or units. Yet, it is impossible to build such systems without defects. As the size of a system increases, it is more likely to develop faults both in the manufacturing process as well as during the operation period. Testing of such systems becomes extremely difficult owing to their large sizes. First, the complexity of test generation for such large systems is overwhelming. Second, the application of test data, as well as the observation and analysis of test responses is extremely difficult and costly, even if test data for the same can be generated. This problem may be

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further aggravated by the likely geographical distribution of units. Testing of such systems based on the traditional stimuli-supplying and responses-observing philosophy has become virtually impossible. In 1967, Preparata, Metze, and Chien [26] proposed a model and a framework, called System-Level Diagnosis, for addressing this problem. In the more than four decades following this pioneering work, several issues arising from the application of this framework have been investigated and resolved. Many of these results have profound theoretical and practical implications. Most of the recent research efforts in system-level diagnosis have focused on enhancing the applicability of system-level diagnosis based approaches to practical scenarios. In particular, the focus has been on:

1. Probabilistic diagnosis and application to VLSI testing.
2. On-line distributed diagnosis and application to the diagnosis of a networked cluster of workstations.

Examples of important advances in system-level diagnosis and applications may be found in [4–11,14,16,17,19,20,26,29].

The focus of this paper is on system diagnosis, which involves locating faulty processors. One of the most popular models in dealing with this problem is the *PMC model*, which was proposed by Preparata, Metze, and Chien [26] in 1967. In the PMC model, a test involves a pair of adjacent processors: the testing and the tested processor. It is also assumed that a test result is reliable if and only if the testing processor is not faulty. Since the introduction of the PMC model, many of its variants have been proposed. Among them, two are particularly relevant in our context: the symmetric comparison model of Chwa and Hakimi [5] and the asymmetric comparison model of Malek [25]. The two models assume the existence of a central observer that collects information about comparisons and then performs a diagnosis of the system. The difference between the two models lies in the different assumptions about the comparison results for two faulty processors. In the symmetric model, it is assumed that the outputs of two compared processors may be the same if they are both faulty while in the asymmetric model it is assumed that the outputs of two such processors are always different. Since for a complex computation task, identical errors for two faulty processors are rare, the asymmetric comparison model is more realistic.

Another model, proposed by Maeng and Malek [24], the MM model, assumes that comparisons are executed by the processors themselves (processors adjacent to both of the two compared ones) and only comparison results are sent to the central observer, which then completes the diagnosis of the system. Maeng and Malek [24] also presented a special case of the MM model, called the MM* model, in which a processor executes comparisons for any pair of its neighboring processors. MM* model is the diagnosis model studied in this paper. Let us describe this model in detail. A graph $G = (V(G), E(G))$ is used to represent a system where each vertex represents a processor and each edge represents a link. Assign a task to each vertex. The vertex w is a comparator of a pair of processors $\{u, v\}$ if $(u, w) \in E(G)$ and $(v, w) \in E(G)$. The outcome of this comparison is denoted by $\sigma((u, v)_w)$ where

$$\sigma((u,v)_w) = \begin{cases} 0, & \text{if } \{u,v,w\} \cap F = \emptyset \\ 1, & \text{if } w \notin F \text{ and } \{u,v\} \cap F \neq \emptyset \\ 0 \text{ or } 1, & \text{if } w \in F \end{cases}$$

where F is the set of faulty processors.

The set of all comparison outcomes is called a syndrome σ of the system. For a given syndrome σ, a subset of vertices $F \subseteq V(G)$ is said to be *consistent* with σ if syndrome σ can be produced when the faulty set of G is F. The comparison result is 0 or 1 when the comparison is performed by a faulty comparator. Therefore, on one hand, a faulty set F may produce a number of different syndromes. On the other hand, different faulty sets may produce the same syndrome. Define $\sigma_F = \{\sigma \mid F \text{ is consistent with } \sigma\}$. Two distinct sets F_1, F_2 belonging to $V(G)$ are said to be *indistinguishable* if $\sigma_{F_1} \cap \sigma_{F_2} \neq \emptyset$; otherwise, F_1, F_2 are said to be *distinguishable*. A system is said to be *t-diagnosable* if, given a syndrome σ, there is a unique set of faulty vertices that is consistent with σ while the number of faulty vertices does not exceed t. The *t-diagnosability problem* is to determine the largest value of t for which a system G is t-diagnosable. Considering classical measures of diagnosability for multiprocessor systems under the comparison model, if all neighbors of a processor v are simultaneously faulty, it is impossible to determine whether the processor v is fault-free or faulty. Therefore, the diagnosability of a system is limited by its minimum vertex degree. For practical systems, the probability that all neighbors of a vertex are simultaneously faulty is very low. Owing to this reason, Lai et al. [22] proposed a new measure of diagnosability as described in the following. A fault set $F \subset V(G)$ is called a *conditional faulty set* if $N_G(v) \nsubseteq F$ for any vertex $v \in V(G)$, where $N_G(v)$ is the set of neighbors of v in G. Two distinct conditional faulty sets F_1, F_2 belonging to $V(G)$ are said to be an indistinguishable conditional pair if $\sigma_{F_1} \cap \sigma_{F_2} \neq \emptyset$; otherwise, F_1, F_2 are said to be a *distinguishable conditional pair*. A system $G = (V(G), E(G))$ is *conditionally t-diagnosable* if any pair of conditional faulty sets F_1, F_2 with $F_1 \neq F_2$, $|F_1| < t$ and $|F_2| < t$ are distinguishable. The *conditional diagnosability* of a system G, denoted as $t_c(G)$, is defined to be the maximum value of t such that G is conditionally t-diagnosable.

A multiprocessor system can be modeled by an undirected simple graph with nodes and links modeled as vertices and edges, respectively. The graph used to model the multiprocessor system is called the interconnection network of the multiprocessor system. Choosing an appropriate interconnection network is important for a system's design and maintenance. Therefore, the analysis of the properties of interconnection networks is an important research topic in high-performance computing.

The hypercube [27] is one of the most popular interconnection networks, and many of its properties have been studied in the literature. Crossed cubes [12], Möbius cube [23], and twisted cubes [15] are several variations of hypercubes. These variants of hypercubes preserve many of the good properties of hypercubes such as high symmetry. At the same time, their diameter is about a half of a hypercube of the same size. Thus, these interconnection networks are regarded as good

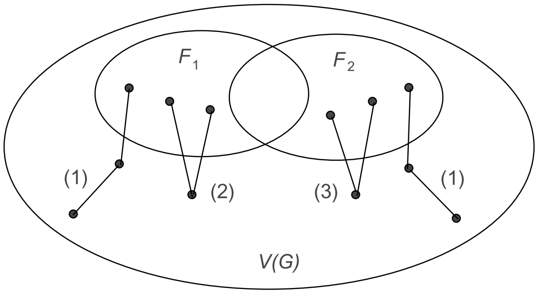

Fig. 1. Illustration for Theorem 1.

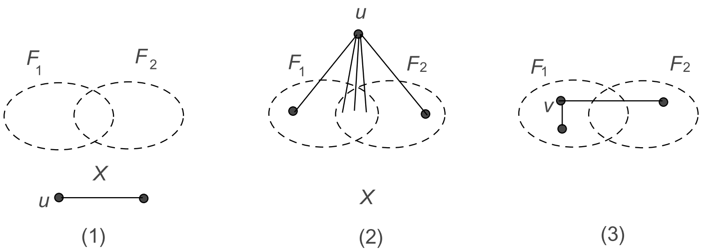

Fig. 2. Illustration for Theorem 2.

alternatives to hypercubes and many of their properties have been explored [1–3,13,21]. The matching composition network (MCN) defined next is a class of interconnection networks that includes a hypercube and its above mentioned variants. Let G_1 and G_2 be two graphs with the same number of vertices. Let M be an arbitrary perfect matching between the vertices of G_1 and G_2; i.e., M is a set of edges with one endpoint in G_1 and the other endpoint in G_2 and $|M| = |V(G_1)| = |V(G_2)|$. An MCN is denoted by $G(G_1, G_2; M)$ with the vertex set $V(G(G_1, G_2; M)) = V(G_1) \cup V(G_2)$ and the edge set $E(G(G_1, G_2; M)) = E(G_1) \cup E(G_2) \cup M$. Exploring the properties of MCN will be of help in discovering many of the properties of the hypercube and its variants.

In [31], Yang studied the conditional diagnosability of a class of MCN under the MM* model. In this paper, we extend the result to a larger class of networks.

The rest of this paper is organized as follows. Section 2 introduces the definitions and known results about the comparison model and matching composition networks. Section 3 focuses on the conditional diagnosability of some matching composition networks. Finally, the conclusion of the paper is presented in Section 4.

2. Preliminaries

Sengupta and Dahbura [28] give a necessary and sufficient condition for a pair of sets being distinguishable under the comparison model.

Theorem 1 *([28]). Let $G = (V(G), E(G))$ be a graph. For any two vertex subsets F_1, F_2 where $F_1, F_2 \subset V(G)$ and $F_1 \neq F_2$, F_1 and F_2 are distinguishable under the comparison model if and only if any one of the following conditions is satisfied (see Fig. 1):*

(1) There exists $\{i, k\} \subseteq V(G) - F_1 - F_2$ and $j \in (F_1 - F_2) \cup (F_2 - F_1)$ such that $(i, k) \in E(G)$ and $(j, k) \in E(G)$;

(2) There exists $\{i, j\} \in F_1 - F_2$ and $k \in V(G) - F_1 - F_2$ such that $(i, k) \in E(G)$ and $(j, k) \in E(G)$;

(3) There exists $\{i, j\} \in F_2 - F_1$ and $k \in V(G) - F_1 - F_2$ such that $(i, k) \in E(G)$ and $(j, k) \in E(G)$.

For conditional faulty sets, Hsu et al. [18] have given three necessary conditions for two sets to be conditionally distinguishable. Let (F_1, F_2) be an indistinguishable conditional pair of G where $G = (V(G), E(G))$ is a connected graph. Let $X = V(G) - (F_1 \cup F_2)$ and $F_1 \triangle F_2 = (F_1 - F_2) \cup (F_2 - F_1)$. If $u \in X$ and $N_G(u) \cap X \neq \emptyset$, then $N_G(u) \cap (F_1 \triangle F_2) = \emptyset$ by condition (1) in Theorem 1. If $u \in X$ and $N_G(u) \cap X = \emptyset$, then $|N_G(u) \cap (F_1 - F_2)| = 1$ and $|N_G(u) \cap (F_2 - F_1)| = 1$ by condition (2) in Theorem 1. If $v \in F_1 \triangle F_2$ and $N_G(v) \cap X = \emptyset$, then $|N_G(v) \cap (F_1 - F_2)| \geq 1$ and $|N_G(v) \cap (F_2 - F_1)| \geq 1$ since F_1 and F_2 are conditional faulty sets. Hsu et al. [18] formalized this fact as below.

Theorem 2 *([18]). Let $G = (V(G), E(G))$ be a connected graph and let $F_1, F_2 \subset V(G)$ be an indistinguishable conditional pair. Let $X = V(G) - (F_1 \cup F_2)$. The following three conditions hold (see Fig. 2):*

(1) $|N_G(u) \cap (F_1 \triangle F_2)| = 0$ for $u \in X$ and $N_G(u) \cap X \neq \emptyset$;

(2) $|N_G(u) \cap (F_1 - F_2)| = 1$ and $|N_G(u) \cap (F_2 - F_1)| = 1$ for $u \in X$ and $N_G(u) \cap X = \emptyset$;

(3) $|N_G(v) \cap (F_1 - F_2)| \geq 1$ and $|N_G(v) \cap (F_2 - F_1)| \geq 1$ for $v \in F_1 \triangle F_2$ and $N_G(v) \cap X = \emptyset$.

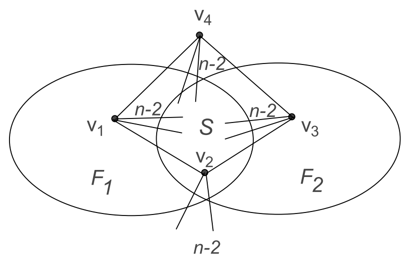

Fig. 3. Illustration for $t_c(Q_n) \leq 3n-3-C(G)$.

Next, we shall introduce some notations and known results about matching composition networks that will be used in the proof of our main result.

Let $G=(V(G), E(G))$ be a graph. For a vertex set $U \subseteq V(G)$, let $N_G(U)=\bigcup_{u\in U} N_G(u)\setminus U$ and $N_G[U]=\bigcup_{u\in U} N_G(u)\bigcup U$. For brevity, $N_G(\{u,v\})$ and $N_G[\{u,v\}]$ are represented as $N_G(u,v)$ and $N_G[u,v]$, respectively. If $|N_G(u)|=k$ for every vertex u in G, then G is *k-regular*. If $N_G(u)\cap N_G(v)=\emptyset$ for any edge (u,v) in G, then G is *triangle-free*. For $u,v\in V(G)$, let $C(G;u,v)=|\{w : w\in N_G(u) \text{ and } w\in N_G(v)\}|$, and $C(G)=\max\{C(G;u,v) : u,v\in V(G)\}$. The *connectivity*, denoted by $\kappa(G)$, is the minimum number of vertices whose removal results in a disconnected or a trivial graph. The *conditional connectivity*, denoted by $\kappa_c(G)$, is the minimum cardinality of a conditional vertex cut which means $\kappa_c(G)=\min\{|F| \mid G-F$ is disconnected and $N_G(u)\nsubseteq F$ for every vertex u of $G\}$. We use $\kappa_1(G)$ to denote $\min\{\kappa(G-N_G[u]) \mid u\in V(G)\}$. In addition, we use $\kappa_2(G)$ to denote $\min\{\kappa(G-N_G[u,v]) \mid u,v\in V(G),\ (u,v)\in E(G)\}$. A graph G is said to be *t-good-connected* if the following properties hold:

1. G is t-connected with $t\geq 1$;
2. $\kappa_c(G)\geq 2t-2\geq 1$;
3. $\kappa_1(G)\geq t-C(G)\geq 1$; and
4. $\kappa_2(G)\geq t-1-C(G)\geq 1$.

By the definition of $C(G)$, we obtain the following lemma directly.

Lemma 3. *Suppose $G=(G_1,G_2;M)$ is a matching composition network such that $C(G_1)\geq 2$ and $C(G_2)\geq 2$. Then $C(G)=\max\{C(G_1),C(G_2)\}\geq 2$.*

The following lemma is given in [30], which shows the relationship between two graphs and their matching composition network.

Lemma 4 *([30]). Suppose $G=(G_1,G_2;M)$ is a matching composition network where G_1 and G_2 are $(n-1)$-regular, $(n-1)$-connected, and triangle-free, then G is an n-regular, n-connected, and triangle-free network.*

Yang obtains the following result for the matching composition network.

Theorem 5 *([31]). Suppose that $G=G(G_1,G_2;M)$ is a matching composition network, where G_1 and G_2 are $(n-1)$-regular, $(n-1)$-good-connected, and triangle-free graphs with $n\geq 5$, $|V(G_1)|=|V(G_2)|=N\geq 3n$, and $C(G_1)=C(G_2)\geq 2$. Then, G is conditionally $(3n-3-C(G))$-diagnosable.*

Obviously, a graph that is t-good connected must be t-connected. In this paper, we extend the result to more networks by weakening the conditions from $(n-1)$-good-connected to $(n-1)$-connected.

Theorem 6. *Suppose $G=(G_1,G_2;M)$ is a matching composition network where G_1 and G_2 are $(n-1)$-regular, $(n-1)$-connected, and triangle-free networks with order no less than $3n+2-C(G)$ for $C(G)\geq 2$. Then, G is conditionally $(3n-3-C(G))$-diagnosable.*

3. Conditional diagnosability of a class of matching composition networks under the comparison model

In the following, we show our main result. Let $G=G(G_1,G_2;M)$ be a matching composition network, where G_1 and G_2 are $(n-1)$-regular, $(n-1)$-connected, and triangle-free graphs with $n\geq 5$ and $C(G_1)\geq 2$, $C(G_2)\geq 2$. Then $C(G)=\max\{C(G_1),\ C(G_2)\}\geq 2$. First, we provide an example to show that the conditional diagnosability of a graph G is less than $3n-3-C(G)$ under the comparison model. Since G is triangle free and $C(G)\geq 2$, we can find the shortest cycle, $C=\langle v_1,v_2,v_3,v_4,v_1\rangle$ of length 4 in G such that $C(G;v_1,v_3)=C(G)$, which is shown in Fig. 3. Let $F_1=N_G(\{v_1,v_3,v_4\})\cup\{v_1\}$ and $F_2=N_G(\{v_1,v_3,v_4\})\cup\{v_3\}$. It is easy to prove that $|F_1|=|F_2|=3n-2-C(G)$ and $|F_1\cap F_2|=3n-3-C(G)$. For any vertex in $V(G)$, there are at most $n-1$ neighbors in F_1 or in F_2 for $C(G)\geq 2$ and $n\geq 5$. Then, F_1 and F_2 are two conditional faulty sets. By Theorem 1, F_1 and F_2 are indistinguishable. Therefore, G is not conditionally $(3n-2-C(G))$-diagnosable and $t_c(Q_n) \le 3n-3-C(G)$.

Owning to the above discussion, we obtain Lemma 7 as follows.

Lemma 7. *Suppose G_1 and G_2 are $(n-1)$-regular, $(n-1)$-connected, and triangle-free networks with order no less than $3n+2-C(G)$ for $n \ge 5$ and $C(G_1) \ge 2$, $C(G_2) \ge 2$. Let $G = (G_1, G_2; M)$ be a matching composition network. Then, $t_c(G) \le 3n-3-C(G)$.*

Next, we prove that $|F_1| \ge 3n-2-C(G)$ or $|F_2| \ge 3n-2-C(G)$ if (F_1, F_2) is an indistinguishable conditional pair in a special class of MCN.

Lemma 8. *Suppose G_1 and G_2 are $(n-1)$-regular, $(n-1)$-connected, and triangle-free networks with order no less than $3n+2-C(G)$ for $n \ge 5$ and $C(G_1) \ge 2$, $C(G_2) \ge 2$. Let $G = (G_1, G_2; M)$ be a matching composition network. If F_1, $F_2 \subset V(G)$, $F_1 \ne F_2$, is an indistinguishable conditional pair of G under the comparison model, then either $|F_1| \ge 3n-2-C(G)$ or $|F_2| \ge 3n-2-C(G)$.*

Proof. By Lemma 3 and Lemma 4, we know that G is an n-regular, n-connected, and triangle-free network with $C(G) = \max\{C(G_1),\ C(G_2)\} \ge 2$.

Let $S = F_1 \cap F_2$. Further, let C be a component in $G - S$ such that $V(C) \cap (F_1 \triangle F_2) \ne \emptyset$. Let D be the induced graph by the vertex set $V(C) \setminus (F_1 \triangle F_2)$. Then $E(D) = \emptyset$ by condition (1) of Theorem 2. To prove the result, we only need to show that $|S| + \lceil \frac{|V(C)|-|V(D)|}{2} \rceil \ge 3n-2-C(G)$. Let $S_1 = V(G_1) \cap S$ and $S_2 = V(G_2) \cap S$.

Next, we prove the result for the following two cases:

Case 1 Either $V(C) \subseteq V(G_1 - S_1)$ or $V(C) \subseteq V(G_2 - S_2)$.

Assume without loss of generality that $V(C) \subseteq V(G_1 - S_1)$. As $V(C) \subseteq V(G_1 - S_1)$, we have $N_{G_2}(C) \subseteq S_2$ and $|V(C)| = |N_{G_2}(V(C))|$. Thus, $|S_2| \ge |V(C)|$. If $V(D) = \emptyset$, then the degree of every vertex in C is greater than 1 by condition 3 of Theorem 2, which implies that there exists a path with length 2 in C. If $V(D) \ne \emptyset$, then the vertex in D has a neighbor in $F_1 - F_2$ and a neighbor in $F_2 - F_1$, which implies there exists a path with length 2 in C. Let $P_3 = \langle x_1, x_2, x_3 \rangle$ be a path with length 2 in C such that $\{(x_1, x_2),\ (x_2, x_3)\} \subseteq E(C)$. Since G_1 is triangle-free, x_1 has a neighbor set X_1 with order $n-2$ in $V(G_1) \backslash V(P_3)$; x_2 has a neighbor set X_2 with order $n-3$ in $V(G_1) \backslash \{V(P_3) \cup X_1\}$; and x_3 has a neighbor set X_3 with order at least $n-1-C(G)$ in $V(G_1) \backslash \{V(P_3) \cup X_1 \cup X_2\}$. We have $|N_{G_1}(\{x_1, x_2, x_3\})| = |X_1| + |X_2| + |X_3| \ge 3n-6-C(G)$. Since $V(P_3) \subseteq V(C) \subseteq V(G_1) \backslash S_1$ and C is a component of $G - S$, we have $N_{G_1}(\{x_1, x_2, x_3\}) \subseteq S_1 \cup \{V(C) \backslash V(P_3)\}$, which implies $|S_1 \cup \{V(C) \backslash V(P_3)\}| \ge |N_{G_1}(P_3)|$. Thus, we have

$$\begin{aligned}
|S| + \lceil \frac{|V(C)|-|V(D)|}{2} \rceil &= |S_1| + |S_2| + \lceil \frac{|V(C)|-|V(D)|}{2} \rceil \\
&\ge |S_1| + |V(C)| + \lceil \frac{|V(C)|-|V(D)|}{2} \rceil \\
&= |S_1| + |V(C) \backslash V(P_3)| + |V(P_3)| + \lceil \frac{|V(C)|-|V(D)|}{2} \rceil \\
&\ge |N_{G_1}(P_3)| + |V(P_3)| + \lceil \frac{|V(C)|-|V(D)|}{2} \rceil \\
&\ge (3n-6-C(G)) + 3 + 1 \\
&\ge 3n-2-C(G).
\end{aligned}$$

Case 2 $V(C) \cap V(G_1 - S_1) \ne \emptyset$ and $V(C) \cap V(G_2 - S_2) \ne \emptyset$.

In this case, let C_1 be the subgraph induced by the vertex set $V(C) \cap V(G_1) \backslash S_1$ and C_2 be the subgraph induced by the vertex set $V(C) \cap V(G_2) \backslash S_2$. We have $N_{G_2}(V(C_1)) \subseteq S_2 \cup V(C_2)$ and $N_{G_1}(V(C_2)) \subseteq S_1 \cup V(C_1)$. Then, $|V(C_1)| \le |S_2| + |V(C_2)|$ and $|V(C_2)| \le |S_1| + |V(C_1)|$.

If $|S| \ge 3n-3-C(G)$, then $|S| + \lceil \frac{|V(C)|-|V(D)|}{2} \rceil \ge (3n-3-C(G)) + 1 = 3n-2-C(G)$, which proves the result.

We now suppose that $|S| \le 3n-4-C(G)$ and need to further divide the proof into the following three subcases. We use $E(D, S)$ to denote the edge set between D and S. Then, $|E(D, S)| = |V(D)| \times (n-2) \le n|S|$ by condition 2 of Theorem 2, which implies that

$$\begin{aligned}
|V(D)| &\le |S| + \lfloor \frac{2 \times |S|}{n-2} \rfloor \\
&\le |S| + \lfloor \frac{2 \times (3n-4-C(G))}{n-2} \rfloor \\
&\le |S| + \lfloor \frac{6(n-2)+4-2C(G)}{n-2} \rfloor \\
&= |S| + 6.
\end{aligned}$$

Subcase 2.1 Both $G_1 - S_1$ and $G_2 - S_2$ are connected.

In this subcase, we have

$$\begin{aligned}|S| + \lceil\frac{|V(C)| - |V(D)|}{2}\rceil &\geq |S| + \lceil\frac{|V(C)| - |S| - 6}{2}\rceil\\ &\geq \lceil\frac{|V(C)| + |S| - 6}{2}\rceil\\ &\geq \lceil\frac{2 \times (3n + 2 - C(G)) - 6}{2}\rceil\\ &\geq 3n - 2 - C(G).\end{aligned}$$

Subcase 2.2 One and only one of $G_1 - S_1$ and $G_2 - S_2$ is connected.

Assume without loss of generality that $G_1 - S_1$ is connected and $G_2 - S_2$ is disconnected. Then, $|S_1| + |V(C_1)| = |V(G_1)| \geq 3n + 2 - C(G)$, $|S_2| \geq n - 1$ and $|S_1| \leq 2n - 5 - C(G)$ for $|S| \leq 3n - 4 - C(G)$.

If $|V(C_2)| + |S_2| \geq 3n - 1 - C(G)$, then

$$\begin{aligned}|S| + \lceil\frac{|V(C)| - |V(D)|}{2}\rceil &= (|S_1| + |S_2|) + \lceil\frac{|V(C_1)| + |V(C_2)| - |V(D)|}{2}\rceil\\ &\geq \lceil\frac{2 \times (|S_1| + |S_2|) + (|V(C_1)| + |V(C_2)|) - |V(D)|}{2}\rceil\\ &\geq \lceil\frac{(|S_1| + |V(C_1)|) + (|S_2| + |V(C_2)|) + (|S| - |V(D)|)}{2}\rceil\\ &\geq \lceil\frac{(3n + 2 - C(G)) + (3n - 1 - C(G)) + [|S| - (|S| + 6)]}{2}\rceil\\ &\geq \lceil\frac{6n - 5 - 2C(G)}{2}\rceil\\ &= 3n - 2 - C(G).\end{aligned}$$

Now, suppose that $|V(C_2)| + |S_2| \leq 3n - 2 - C(G)$, which means that $V(G_2) \setminus \{V(C_2) \cup S_2\} \neq \emptyset$ for $|V(G_2)| \geq 3n + 2 - C(G)$. Since G_2 is $(n-1)$-connected, there exists at least $n - 1$ edges joining $V(G_2) \setminus \{V(C_2) \cup S_2\}$ and S_2. Then, $|E(D, S)| = |V(D)| \times (n - 2) \leq n|S| - (n - 1)$ by condition 2 of Theorem 2, which implies that

$$\begin{aligned}|V(D)| &\leq |S| + \lfloor\frac{2 \times |S| - (n - 1)}{n - 2}\rfloor\\ &\leq |S| + \lfloor\frac{2 \times (3n - 4 - C(G)) - (n - 1)}{n - 2}\rfloor\\ &\leq |S| + \lfloor\frac{5(n - 2) + 3 - 2C(G)}{n - 2}\rfloor\\ &= |S| + 4.\end{aligned}$$

Since $G_1 - S_1$ is connected, $N_{G_1}(V(G_2) \setminus \{V(C_2) \cup S_2\}) \subseteq S_1$. We get $|V(G_2) \setminus \{V(C_2) \cup S_2\}| = |N_{G_1}(V(G_2) \setminus \{V(C_2) \cup S_2\})| \leq |S_1|$. Thus,

$$\begin{aligned}3n - 4 - C(G) &\geq |S|\\ &= |S_1| + |S_2|\\ &\geq (|V(G_2)| - |S_2| - |V(C_2)|) + |S_2|\\ &= |V(G_2)| - |V(C_2)|\\ &\geq 3n + 2 - C(G) - |V(C_2)|,\end{aligned}$$

which implies that $|V(C_2)| \geq 6$.

If there exists a path P_3 with length two in C_2, then $|S_2| + |V(C_2)| \geq |N_{G_2}(P_3)| + |V(P_3)| \geq 3n - 3 - C(G)$ by the same discussion as in case 1. Then,

$$\begin{aligned}|S| + \lceil\frac{|V(C)| - |V(D)|}{2}\rceil &= (|S_1| + |S_2|) + \lceil\frac{|V(C_1)| + |V(C_2)| - |V(D)|}{2}\rceil\\ &\geq \lceil\frac{2 \times (|S_1| + |S_2|) + (|V(C_1)| + |V(C_2)|) - |V(D)|}{2}\rceil\\ &\geq \lceil\frac{(|S_1| + |V(C_1)|) + (|S_2| + |V(C_2)|) + (|S| - |V(D)|)}{2}\rceil\end{aligned}$$

$$\begin{aligned}
&\geq \lceil \frac{(3n+2-C(G))+(3n-3-C(G))+[|S|-(|S|+4)]}{2} \rceil \\
&\geq \lceil \frac{6n-5-2C(G)}{2} \rceil \\
&= 3n-2-C(G).
\end{aligned}$$

Otherwise, for an arbitrary vertex in $V(C_2)$, there is at most one neighbor in C_2 and at least $n-2$ neighbors in S_2. If $E(C_2)=\emptyset$, then choose $\{u,v\} \subseteq V(C_2)$ arbitrarily. The vertex u has a neighbor set Z_1 in S_2 with order $n-1$ and v has a neighbor set Z_2 in $S_2 - Z_1$ with order $n-1-C(G)$. Then, $|S_2| \geq |Z_1|+|Z_2| \geq 2n-2-C(G)$. Otherwise, there exists an edge $(u,v) \in E(G)$. Then, the vertex u has a neighbor set Z_1 in S_2 with order $n-2$ and v has a neighbor set Z_2 in $S_2 - Z_2$ with order $n-2$. Then, $|S_2| \geq |Z_1|+|Z_2| \geq 2n-4 \geq 2n-2-C(G)$. For arbitrary vertex $u \in V(C_2) \cap V(D)$, then $N_{G_1}(u) \nsubseteq S_1$ and $N_{G_1}(u) \subseteq V(C_1) \setminus V(D)$ by condition 2 of Theorem 2. Thus,

$$\begin{aligned}
|V(C)|-|V(D)| &= |V(C_1) \setminus V(D)| + |V(C_2) \setminus V(D)| \\
&\geq |V(C_2) \cap V(D)| + |V(C_2) \setminus V(D)| \\
&= |V(C_2)| \\
&\geq 6.
\end{aligned}$$

If $|S| \geq 3n-5-C(G)$, then

$$\begin{aligned}
|S| + \lceil \frac{|V(C)|-|V(D)|}{2} \rceil &\geq (3n-5-C(G))+3 \\
&= 3n-2-C(G).
\end{aligned}$$

If $|S| \leq 3n-6-C(G)$, then $|S_1| = |S|-|S_2| \leq (3n-6-C(G))-(2n-2-C(G)) = n-4$. For each vertex in $V(C_1) \cap V(D)$, there are at least $n-3$ neighbors in S_1 by condition 2 of Theorem 2. Therefore, we claim $V(C_1) \cap V(D) = \emptyset$.

$$\begin{aligned}
|S| + \lceil \frac{|V(C)|-|V(D)|}{2} \rceil &= |S_2| + \lceil \frac{(|S_1|+|V(C_1)|)+|S_1|+(|V(C_2)|-|V(D)|)}{2} \rceil \\
&\geq 2n-2-C(G) + \lceil \frac{3n+2-C(G)}{2} \rceil \\
&\geq 2n-2-C(G) + \lceil \frac{2n+(n+2-C(G))}{2} \rceil \\
&\geq 3n-2-C(G).
\end{aligned}$$

Subcase 2.3 Both $G_1 - S_1$ and $G_2 - S_2$ are disconnected.

In this subcase, we have $n-1 \leq |S_1| \leq 2n-3-C(G)$ and $n-1 \leq |S_2| \leq 2n-3-C(G)$. We discuss this subcase for the following cases.

Subcase 2.3.1 $|V(D) \cap V(C_1)| \geq 2$ and $|V(D) \cap V(C_2)| \geq 2$.

For two arbitrary vertices $\{u,v\} \subseteq V(C_1)$, we have $|N_{G_1}\{u,v\}| \geq 2n-2-C(G) > |S_1|$. Thus, there exists an edge $e = (u,v)$ in C_1. For $|N_{G_1}(\{u,v\})| = 2(n-2)$ and $|S_1| \leq 2n-3-C(G)$, there exists a path $P_3 = \langle u,v,w \rangle$ of length 2 in C_1 with $\{(u,v),(v,w)\} \subseteq E(G_1)$ and $(u,w) \notin E(G_2)$. By the similar discussion as case 1, $|V(C_1)|+|S_1| \geq 3n-3-C(G)$ and $|V(C_2)|+|S_2| \geq 3n-3-C(G)$.

Subcase 2.3.1A At most one vertex u in $V(C_1) \cap V(D)$ has two neighbors in $V(C_1) \setminus V(D)$ or at most one vertex u in $V(C_2) \cap V(D)$ has two neighbors in $V(C_2) \setminus V(D)$.

Without loss of generality, we assume at most one vertex u in $V(C_1) \cap V(D)$ has two neighbors in $V(C_1) \setminus V(D)$. In this subcase, $N_{G_2}(v) \subseteq V(C_2) \setminus V(D)$ for each vertex v in $V(C_1) \cap V(D) \setminus \{u\}$, which means that $|V(C_1) \cap V(D)| - 1 \leq |V(C_2) \setminus V(D)|$.

Then,

$$\begin{aligned}
&|S| + \lceil \frac{|V(C)|-|V(D)|}{2} \rceil \\
&= (|S_1|+|S_2|) + \lceil \frac{|V(C_1) \setminus V(D)| + |V(C_2) \setminus V(D)|}{2} \rceil \\
&\geq |S_2| + \lceil \frac{[|V(C_1) \setminus V(D)| + (|V(C_1) \cap V(D)|-1) + |S_1|] + |S_1|}{2} \rceil \\
&\geq (n-1) + \lceil \frac{(|V(C_1)|+|S_1|-1)+(n-1)}{2} \rceil \\
&\geq (n-1) + \lceil \frac{(3n-4-C(G))+(n-1)}{2} \rceil \\
&\geq 3n-2-C(G).
\end{aligned}$$

Subcase 2.3.1B There exists at least two vertices in $V(C_1) \cap V(D)$ that have two neighbors in $V(C_1) \setminus V(D)$ and there exists at least two vertices in $V(C_2) \cap V(D)$ that have two neighbors in $V(C_2) \setminus V(D)$.

In this subcase, we have $|V(C_1) \setminus V(D)| \geq 2$ and $|V(C_2) \setminus V(D)| \geq 2$. Choose u and v in $V(C_1) \cap V(D)$ such that $|N_{C_1}(u)| = |N_{C_1}(v)| = 2$. Then, the vertex u has a neighbor set Z_1 in S_1 with order $n-3$ and v has a neighbor set Z_2 in $S_1 - Z_1$ with order $n-1-C(G)$. Then, $|S_1| \geq |Z_1| + |Z_2| \geq 2n-4-C(G)$. Along the same lines, $|S_2| \geq 2n-4-C(G)$.

If $C(G) \geq n-2$, then

$$\begin{aligned}
&|S| + \lceil \frac{|V(C)| - |V(D)|}{2} \rceil \\
&\geq (n-1) + (n-1) + 2 \\
&\geq 3n-2-C(G).
\end{aligned}$$

If $C(G) \leq n-4$, then,

$$\begin{aligned}
&|S| + \lceil \frac{|V(C)| - |V(D)|}{2} \rceil \\
&= (|S_1| + |S_2|) + \frac{|V(C_1) \setminus V(D)| + |V(C_2) \setminus V(D)|}{2} \rceil \\
&\geq (2n-4-C(G)) + (2n-4-C(G)) + 2 \\
&\geq (3n-2-C(G)) + (n-4-C(G)) \\
&\geq 3n-2-C(G).
\end{aligned}$$

If $C(G) = n-3$ and $|V(C_1) \setminus V(D)| + |V(C_2) \setminus V(D)| \geq 5$,

$$\begin{aligned}
&|S| + \lceil \frac{|V(C)| - |V(D)|}{2} \rceil \\
&\geq (n-1) + (n-1) + 3 \\
&= 3n-2-C(G).
\end{aligned}$$

If $C(G) = n-3$ and $|S_1| + |S_2| \geq 2n-1$, then

$$\begin{aligned}
&|S| + \lceil \frac{|V(C)| - |V(D)|}{2} \rceil \\
&\geq (2n-1) + 2 \\
&= 3n-2-C(G).
\end{aligned}$$

Otherwise, $C(G) = n-3$, $|V(C_1) \setminus V(D)| = |V(C_2) \setminus V(D)| = 2$ and $|S_1| = |S_2| = n-1$. Then $|V(C_1) \cap V(D)| \geq n-1 \geq 4$ for $|V(C_1)| + |S_1| \geq 3n-3-C(G)$. Each vertex in $V(C_1) \cap V(D)$ has $n-1$ neighbors in $S_1 \cup V(C_1) \setminus V(D)$. Then, there exists two vertices u and v in $V(C_1) \cap V(D)$ such that $C(G, u, v) = |N_{G_1}(u) \cap N_{G_1}(v)| \geq n-2$ for $|S_1 \cup V(C_1) \setminus V(D)| = (n-1) + 2 = n+1$, a contradiction with $C(G) = n-3$.

Subcase 2.3.2 $|V(C_1) \cap V(D)| \leq 1$ or $|V(C_2) \cap V(D)| \leq 1$. Assume without loss of generality that $|V(C_1) \cap V(D)| \leq 1$.

If $|V(C_1)| = 1$, then $|V(C_2)| \geq 2$ and $V(C_1) \cap D = \emptyset$ by Theorem 2. We obtain $|C_2| + |S_2| \geq 3n-3-C(G)$ by the same discussion as in subcase 2.3.1. Then, we have

$$\begin{aligned}
&|S| + \lceil \frac{|V(C)| - |V(D)|}{2} \rceil \\
&= |S_1| + |S_2| + \lceil \frac{|V(C_1)| + |V(C_2)| - |V(D)|}{2} \rceil \\
&\geq (|V(C_2)| - |V(C_1)|) + |S_2| + 2 \\
&= (|S_2| + |V(C_2)|) + 1 \\
&\geq 3n-2-C(G).
\end{aligned}$$

The second inequality holds for $N_{G_1}(C_2) \subseteq S_1 \cup V(C_1)$.

If $|C_1| \geq 2$, then $|C_1| + |S_1| \geq 3n-3-C(G)$ by the same discussion as above.

Then, we have

$$\begin{aligned}
|S| + \lceil \frac{|V(C)| - |V(D)|}{2} \rceil &= (|S_1| + |S_2|) + \lceil \frac{|V(C_1)| + |V(C_2)| - |V(D)|}{2} \rceil \\
&\geq |S_2| + \lceil \frac{2|S_1| + |V(C)| - |V(D)|}{2} \rceil
\end{aligned}$$

$$\begin{aligned}
&\geq |S_2| + \lceil \frac{(|S_1| + |V(C_1)| - |V(D_1)|) + |S_1| + (|V(C_2)| - |V(D_2)|}{2} \rceil \\
&\geq (n-1) + \lceil \frac{(3n-3-C(G)-1)+(n-1)}{2} \rceil \\
&\geq (n-1) + \lceil \frac{4n-5-C(G)}{2} \rceil \\
&\geq 3n-2-C(G).
\end{aligned}$$

This completes the proof of Lemma 8. □

Now, by applying the above lemmas, we obtain the main result.

Theorem 6. *Suppose $G = (G_1, G_2; M)$ is a matching composition network where G_1 and G_2 are $(n-1)$-regular, $(n-1)$-connected, triangle-free networks with order no less than $3n + 2 - C(G)$ and $C(G) \geq 2$. Then, G is conditionally $(3n - 3 - C(G))$-diagnosable.*

Proof. By Lemma 7, we have $t_c(G) \leq 3n - 3 - C(G)$. By Lemma 8, we have $t_c(G) \geq 3n - 3 - C(G)$. Then, it proves $t_c(G) = 3n - 3 - C(G)$. □

We next highlight the differences between Yang's result [31] and our Theorem 6.

	Yang	Theorem 6
$(n-1)$-regular $(n-1)$-connected triangle free	Required	Required
$(n-1)$-good connected	Required	Not required
$C(G_1) = C(G_2)$	Required	Not required
$C(G_1) > 2$	Not allowed	Allowed
Number of vertices	$\geq 3n$	$\geq 3n$ if $C(G) = 2$ $\geq 3n + 2 - C(G)$ if $C(G) > 2$

In view of the above, we can see that Theorem 6 extends Yang's result by demonstrating the $3n - 3 - C(G)$ conditional diagnosability of a larger class of networks than those determined by Yang.

For the hypercube Q_n, the crossed cube CQ_n, the twisted cube TQ_n, and the Möbius cube MQ_n, all the networks are n-regular, n-connected, and triangle-free matching composition networks composed by two $(n-1)$-dimensional subcubes. Furthermore, all of these networks contain a cycle of length four and every two vertices have at most two common neighbors. Since $|V(Q_{n-1})| = |V(CQ_{n-1})| = |V(TQ_{n-1})| = |V(MQ_{n-1})| = 2^{n-1} \geq 3n$ for $n \geq 5$, we obtain the results of [18,32,33] as corollaries of Theorem 6.

Corollary 10 *(Lai et al. [18]). The conditional diagnosability of hypercube Q_n is $t_c(Q_n) = 3n - 5$ for $n \geq 5$.*

Corollary 11 *(Zhou [32]). The conditional diagnosability of the twisted cube TQ_n is $3n - 5$ for $n \geq 5$.*

Corollary 12 *(Zhou [33]). The conditional diagnosability of the crossed cube CQ_n is $3n - 5$ for $n \geq 5$.*

Corollary 13. *The conditional diagnosability of the Möbius cube MQ_n is $3n - 5$ for $n \geq 5$.*

4. Conclusion

Malek and Maeng proposed the comparison model, which is an attractive tool for fault diagnosis of a system. In applications of system fault diagnosis, the probability that all neighbors of a vertex are faulty simultaneously is very low. For this reason, Lai et al. [22] proposed conditional diagnosability, which is a new measure of diagnosability. In [31] Yang determined a set of sufficient conditions for a network G to be conditionally $(3n - 3 - C(G))$-diagnosable. In this paper we extend this result by determining a larger class of networks that are conditionally $(3n - 3 - C(G))$-diagnosable. Also, the earlier results in [18,32,33] on conditional diagnosability of the hypercube, the cross cube, the twisted cube and the Möbius all become corollaries of our main result. Thus the work in this paper extends the state of the art on the conditional diagnosability of multiprocessor systems.

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