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$(n+p)$-NODE REALIZABILITY OF Y-MATRICES OF $(n+1)$-NODE RESISTANCE n-PORT NETWORKS

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SUMMARY

Sufficient conditions are established for the $(n+p)$-node, $p>2$, realizability of Y-matrices of $(n+1)$-node resistance n-port networks. It is shown that these conditions are a generalization of the previous known results for $p=2$ and $p=n$.

1. INTRODUCTION

The n-port resistance network synthesis problem has long been a subject of research. While the complete solution of the problem seems to be formidable, several aspects of this problem have been investigated during the last two decades or so. Two of the recent contributions include those of References 1 and 2. While in Reference 2 the problem of $(n+2)$-node realizability of Y-matrices of $(n+1)$-node n-port networks is considered, an interesting study of degenerate resistance networks is made in Reference 1.

In this paper we consider the problem of $(n+p)$-node, $p>2$, realizability of Y-matrices of $(n+1)$-node resistive n-port networks. It might appear that if a matrix is realizable by an $(n+1)$-node resistive n-port network, then it should also be realizable by an n-port network having more than $n+1$ nodes. However, such is not the case. In fact, Lempel and Cederbaum[3] have shown that certain extremal Y-matrices can not be realized by networks with more than $(n+1)$ nodes. They have also shown that for each $n \geq 3$, and p, $1 \leq p \leq n$, there exists a realizable nth order Y-matrix which can be realized with any number of nodes between $n+1$ and $n+p$, but not more than $n+p$. For example, the fifth-order extremal Y-matrix

$$\mathbf{Y} = \begin{bmatrix} 5 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is realizable with 6, 7 or 8 nodes but can not be realized with more than 8 nodes. These results of Lempel and Cederbaum motivated the work of Reference 2 in which it was shown that if a matrix is realizable by an $(n+1)$-node resistive n-port network containing no zero conductances, then it can also be realized by an $(n+2)$-node n-port network. It was also shown that for such a matrix a large number of continuously equivalent realizations can be obtained. We continue our pursuit in this paper and establish sufficient conditions for the $(n+p)$-node, $p>2$, realizability of Y-matrices realizable by $(n+1)$node n-port networks. We shall also show that these conditions generalize the previously known conditions for $p=2$ and $p=n$.

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2. $(n+p)$-NODE REALIZABILITY OF Y-MATRICES OF $(n+1)$-NODE n-PORT NETWORKS

We choose, for our discussions, the dominant matrix **Y** with all off-diagonal entries negative. Note that such a matrix is realizable by an $(n+1)$-node n-port network having a star-port configuration. This choice would involve no loss of generality, since the $(n+p)$-node realizability of any other $(n+1)$-node realizable matrix can be tested by first converting it to a dominant matrix with all off-diagonal entries negative and then applying the conditions to be derived in this section.

Let the given matrix **Y** be required to be realized by an n-port network having the port configuration T shown in Figure 1. Let $T_1, T_2, \ldots, T_p$ denote the p connected parts of T. Let T_0 be a tree of N so that T is a subgraph of T_0. $P_{i(k)}$ will denote the kth port in T_i, while i_k and i_0 are respectively the positive and negative terminals of port $P_{i(k)}$. $y_{i(k),j(m)}$ will denote the transfer conductance between ports $P_{i(k)}$ and $P_{j(m)}$.

Network N can be considered as a parallel combination of two n-port networks—the network of departure N_d and the padding n-port network N_p.[4] Properties of N_d and N_p are discussed in Reference 4.

Let $g_{i_kj_m}$ denote the conductance connecting vertices i_k and j_m in N. $(g_{i_kj_m})_d$ and $(g_{i_kj_m})_p$ will denote the corresponding conductances of N_d and N_p respectively. Let further

$$S_{i_kj} = \sum_{m=0}^{n_j} g_{i_kj_m},$$

$$S_{ij} = \sum_{k=0}^{n_i} S_{i_kj}$$

$$= \sum_{m=0}^{n_j} S_{j_mi} \qquad i, j = 1, 2 \ldots p,\ i \neq j \tag{1}$$

$(S_{i_kj})_d$, $(S_{i_kj})_p$, $(S_{ij})_d$ and $(S_{ij})_p$ will denote the corresponding quantities of N_d and N_p respectively. It can be seen that

$$\begin{aligned} g_{i_kj_m} &= (g_{i_kj_m})_d + (g_{i_kj_m})_p, \\ S_{i_kj} &= (S_{i_kj})_d + (S_{i_kj})_p \\ S_{ij} &= (S_{ij})_d + (S_{ij})_p \end{aligned} \tag{2}$$

It is shown in Reference 4 that for any k

$$(S_{i_kj})_d = 0 \quad \text{for all } i \text{ and } j,\ i \neq j$$

Thus we have

$$S_{i_kj} = (S_{i_kj})_p$$

and

$$S_{ij} = (S_{ij})_p \tag{3}$$

Let $k_{i(k),j}$ denote the potential of the set of ports in T_j with respect to the negative reference terminal of port $P_{i(k)}$, when $P_{i(k)}$ is excited with a source of unit voltage and when all the other ports of N are short-circuited. Note that under such a condition, all the ports in T_j will be at the same potential.

Our approach to obtain the conditions on **Y** for its $(n+p)$-node realizability will be a combination of those of References 2, 5 and 6. Thus we assume that the required network N satisfies the following conditions:

(i) For any i and k

$$k_{i(k),j} = k_{i(k),m} \quad \text{for all } j \text{ and } m,\ j \neq m \tag{4}$$

(ii)

$$g_{i_0j_m} = 0 \quad \text{for all } i \text{ and } j,\ j \neq i \text{ and for all } m \neq 0 \tag{5}$$

The first condition means that when $P_{i(k)}$ is excited and all the other ports are short-circuited, no current flows in the conductances connecting vertices in T_j and T_l for all j and l, $j \neq l \neq i$. This is, in fact, a generalization of the approach followed in References 5 and 6. The conditions are necessary to obtain simple conditions for the $(n+p)$-node realizability of $\mathbf{Y}$.

The conductances of N_d can be obtained starting from its cutset admittance matrix $\mathbf{Y}_d$. $\mathbf{Y}_d$ will be as shown below.

$$\mathbf{Y}_d = \left[\begin{array}{c|c} \mathbf{Y} & 0 \\ \hline 0 & 0 \end{array}\right]\begin{array}{l} n \\ p-1 \end{array} \tag{6}$$

In $\mathbf{Y}_d$ the rows and columns of $\mathbf{Y}$ correspond to the branches of T and the remaining rows and columns of $\mathbf{Y}_d$ correspond to the branches of $T_0 - T$. Conductances of N_d do not, of course, depend on the choice of T_0. We can choose T_0 as consisting of T and those branches indicated by dotted lines in Figure 1. Realizing $\mathbf{Y}_d$ by an $(n+p-1)$-port network having the port configuration T_0 will yield the conductances of N_d. These conductances can be easily obtained following the procedure given in Reference 7. They are as follows:

$$(g_{i_k j_m})_d = |y_{i(k),j(m)}|, \qquad \begin{array}{l} i, j = 1, 2, \dots p \\ k = 1, 2, \dots n_i \\ m = 1, 2, \dots n_j \\ m \neq k, \text{ when } i = j \end{array} \tag{7}$$

$$\begin{aligned} (g_{i_k j_0})_d &= -\sum_{m=1}^{n_j} (g_{i_k j_m})_d \\ &= \sum_{m=1}^{n_j} y_{i(k),j(m)} \qquad i, j = 1, 2 \dots, p \\ &\qquad\qquad\qquad\qquad\quad j \neq i \end{aligned} \tag{8}$$

$$\begin{aligned} (g_{i_k i_0})_d &= y_{i(k),i(k)} - \sum_{\substack{m=1 \\ m \neq k}}^{n_i} (g_{i_k i_m})_d \\ &= y_{i(k),i(k)} - \sum_{\substack{m=1 \\ m \neq k}}^{n_i} |y_{i(k),i(m)}| \end{aligned} \tag{9}$$

$$(g_{i_0 j_0})_d = \sum_{m=1}^{n_j} \sum_{k=1}^{n_i} |y_{i(k),j(m)}| \tag{10}$$

The conductances of N_p can be obtained by using the formulae given in Reference 8. These formulae reduce to the following after using equation (4).

$$(g_{i_k j_m})_p = \frac{S_{i_k j} \times S_{j_m i}}{S_{ij}}, \qquad \begin{array}{l} i, j = 1, 2 \dots, p, i \neq j \\ k = 1, 2 \dots, n_i \\ m = 1, 2 \dots, n_j \end{array} \tag{11}$$

$$(g_{i_k i_m})_p = -\sum_{\substack{j=1 \\ j \neq i}}^{p} \frac{S_{i_k j} S_{i_m j}}{S_{ij}} \tag{12}$$

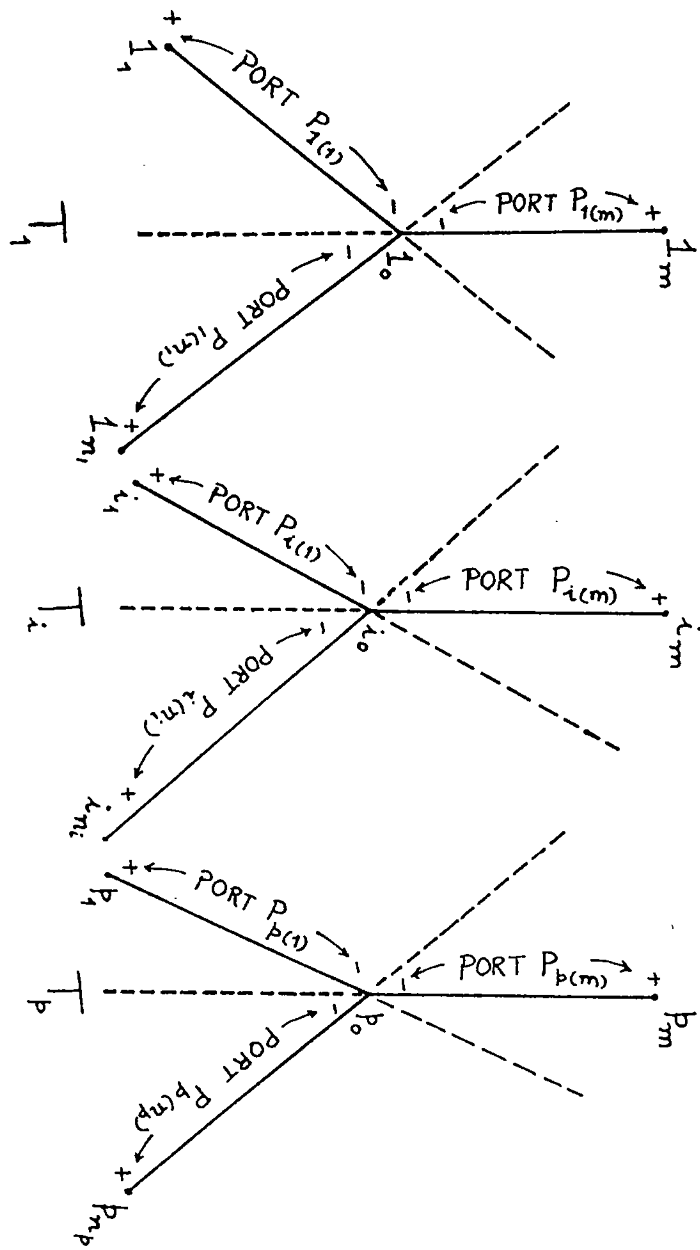

Figure 1. Port-configuration T

It can be shown that under the conditions imposed by equation (4)

$$k_{i(k),j} = \frac{S_{i_k j}}{S_{ij}} \tag{13}$$

We shall next establish the constraints imposed by equations (4) and (5) on the elements of the matrix **Y**.

By equation (5)

$$g_{i_k j_o} = (g_{i_k j_o})_d + (g_{i_k j_o})_p = 0 \tag{14}$$

Using equations (8), (11), and (13) we then get from the above

$$\sum_{m=1}^{n_j} |y_{i(k),j(m)}| = \frac{S_{i_k j} S_{j_o i}}{S_{ij}} = k_{i(k),j} S_{j_o i} = k_{i(k),j} g_{j_o i_o} \tag{15}$$

The last step in the above equation follows from the fact

$$S_{j_o i} = \sum_{m=1}^{n_i} g_{j_o i_m} + g_{j_o i_o}$$

$= g_{j_o i_o}$, since each $g_{j_o i_m} = 0$ by equation (5).

After defining $\Delta_{i(k),j}$ as

$$\Delta_{i(k),j} = \sum_{m=1}^{n_j} |y_{i(k),j(m)}| \tag{16}$$

we get from equation (15)

$$k_{i(k),j} = \frac{\Delta_{i(k),j}}{g_{i_o j_o}} \tag{17}$$

Since by equation (4)

$$k_{i(k),j} = k_{i(k),m} \qquad \text{for all } m \neq j$$

we get from equation (17)

$$\frac{\Delta_{i(k),j}}{g_{i_o j_o}} = \frac{\Delta_{i(k),m}}{g_{i_o m_o}} \qquad i, j, m = 1, 2 \ldots, p, \quad i \neq j \neq m, \quad k = 1, 2 \ldots n_i$$

or

$$\frac{g_{i_o j_o}}{g_{i_o m_o}} = \frac{\Delta_{i(k),j}}{\Delta_{i(k),m}} \tag{18}$$

Since equation (18) has to be true for all k, we get the following condition on the entries of Y:

$$\frac{\Delta_{i(k),j}}{\Delta_{i(k),m}} = \frac{\Delta_{i(r),j}}{\Delta_{i(r),m}} \qquad i, j, m = 1, 2 \ldots, p, \quad i \neq j \neq m, \quad k, r = 1, 2 \ldots, n_i \tag{19}$$

The problem of determining the network N realizing **Y** involves the choice of suitable nonnegative values for $S_{i_k j}$'s and S_{ij}'s so that each $g_{i_k j_m} = (g_{i_k j_m})_d + (g_{i_k j_m})_p$ is nonnegative. We shall show that such a choice is possible if the matrix **Y** satisfies equation (19).

By equation (5), the conductances $g_{i_k j_o}$'s of N are zero. From equations (7) and (11), it can be seen that the conductances $(g_{i_k j_m})_d$'s and $(g_{i_k j_m})_p$'s $i \neq j$, $m \neq 0$ and $k \neq 0$ are nonnegative. Further by equation (10),

$(g_{i_o j_o})_d$'s and by equation (11), $(g_{i_o j_o})_p$'s are nonnegative. Thus all the conductances $g_{i_k j_m}$'s $i \neq j$ are nonnegative as long as $S_{i_k j}$'s and S_{ij}'s are nonnegative.

It can be seen from equations (7), (9) and (12) that all $(g_{i_k i_m})_d$'s are nonnegative while $(g_{i_k i_m})_p$'s are nonpositive. Thus, it is in making the sum of $(g_{i_k i_m})_d$ and $(g_{i_k i_m})_p$ nonnegative that the choice of $S_{i_k j}$'s and S_{ij}'s matters. We shall next proceed to show how such a choice can be made.

From equation (18) we have

$$g_{i_o m_o} = \frac{\Delta_{i(k),m}}{\Delta_{i(k),j}} g_{i_o j_o} \qquad \text{for some } k \tag{20}$$

We can use the above to show that all $g_{i_o j_o}$'s can be expressed in terms of one of them, say, $g_{1_o 2_o}$.

Thus

$$\begin{aligned} g_{i_o j_o} &= \frac{\Delta_{i(k),j}}{\Delta_{i(k),1}} \times g_{1_o i_o}, \qquad \text{for some } k \\ &= \frac{\Delta_{i(k),j}}{\Delta_{i(k),1}} \times \frac{\Delta_{1(m),i}}{\Delta_{1(m),2}} \times g_{1_o 2_o} \quad \text{for some } k \text{ and } m \\ &\triangleq \alpha_{ij} g_{1_o 2_o} \end{aligned}$$

Note that $\alpha_{12} = 1$

It then follows from equation (17) that

$$\begin{aligned} k_{i(k),j} &= \left(\frac{\Delta_{i(k),j}}{\alpha_{ij}}\right) \times \frac{1}{g_{1_o 2_o}} \\ &\triangleq \beta_{i_k j} \times \frac{1}{g_{1_o 2_o}} \end{aligned} \tag{22}$$

Thus all the potential factors $k_{i(k),j}$'s can be expressed in terms of a single parameter, namely, $g_{1_o 2_o}$.

Consider next equation (12). This can be rewritten using equation (13) as

$$\begin{aligned} (g_{i_k i_m})_p &= -\sum_{\substack{j=1 \\ j \neq i}}^{p} k_{i(k),j} k_{i(m),j} S_{ij} \qquad m \neq k \\ &\qquad\qquad\qquad\qquad\qquad\qquad m \neq 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad k \neq 0 \\ (g_{i_k i_o})_p &= \sum_{\substack{j=1 \\ j \neq i}}^{p} k_{i(k),j} S_{i_o j} \\ &= -\sum_{\substack{j=1 \\ j \neq i}}^{p} k_{i(k),j} g_{i_o j_o} \qquad k \neq 0 \end{aligned} \tag{23}$$

Hence evaluation of $(g_{i_k i_m})_p$'s requires an expression for S_{ij}'s in terms of $g_{1_o 2_o}$.

By definition

$$\begin{aligned} S_{ij} &= \sum_{k=0}^{n_i} S_{i_k j} \\ &= \sum_{k=1}^{n_i} S_{i_k j} + S_{i_o j} \\ &= \sum_{k=1}^{n_i} S_{i_k j} + g_{i_o j_o} \end{aligned}$$

Dividing both sides of the above by S_{ij}, we get using equation (13)

$$1 = \sum_{k=1}^{n_i} k_{i(k),j} + \frac{g_{iojo}}{S_{ij}}$$

$$= \sum_{k=1}^{n_i} \frac{\Delta_{i(k),j}}{g_{iojo}} + \frac{g_{iojo}}{S_{ij}} \tag{24}$$

From the above we get S_{ij} as

$$S_{ij} = \frac{g_{iojo}^2}{g_{iojo} - \Delta_{ij}}$$

$$= \frac{\alpha_{ij}^2 g_{1o2o}^2}{\alpha_{ij} g_{1o2o} - \Delta_{ij}} \tag{25}$$

where

$$\Delta_{ij} = \sum_{k=1}^{n_i} \Delta_{i(k),j}$$

Using equations (17) and (25) in equation (23), we get

$$(g_{i_k i_m})_p = -\sum_{j=1}^{p} \frac{\Delta_{i(k),j} \Delta_{i(m),j}}{\alpha_{ij} g_{1o2o} - \Delta_{ij}}, \qquad m \neq k, \quad m \neq 0, \quad k \neq 0 \tag{26}$$

and

$$(g_{i_k i_o})_p = -\sum_{\substack{j=1 \\ j \neq i}}^{p} \Delta_{i(k),j} \tag{27}$$

Consider equations (27) and (9). We get from these

$$g_{i_k i_o} = (g_{i_k i_o})_d + (g_{i_k i_o})_p$$

$$= y_{i(k),i(k)} - \sum_{j=1}^{p} \sum_{\substack{m=1 \\ m \neq k, \text{ when } j=i}}^{n_j} |y_{i(k),j(m)}|$$

= a nonnegative number, since **Y** is dominant.

From equations (26) and (7), we get

$$g_{i_k i_m} = |y_{i(k),i(m)}| - \sum_{\substack{j=1 \\ j \neq i}}^{p} \frac{\Delta_{i(k),j} \Delta_{i(m),j}}{\alpha_{ij} g_{1o2o} - \Delta_{ij}} \tag{28}$$

Nonnegativeness of $g_{i_k i_m}$ requires

$$|y_{i(k),i(m)}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{p} \frac{\Delta_{i(k),j} \Delta_{i(m),j}}{\alpha_{ij} g_{1o2o} - \Delta_{ij}}, \qquad m \neq k \tag{29}$$

Since in the above g_{1o2o} appears in the denominator and all other terms are positive it is always possible to choose a suitable positive value for g_{1o2o} so that inequality (29) is satisfied.

Such a choice for g_{1o2o} will ensure that the conductances of N are nonnegative. This value for g_{1o2o} can be used to evaluate S_{ij}'s (using equation (25)), g_{iojo}'s (using equation (21)), and $(g_{i_k j_m})_p$'s (using equations (11) and (12)). The sum of $(g_{i_k j_m})_p$ and $(g_{i_k j_m})_d$ will then yield a nonnegative $g_{i_k j_m}$.

If for some i and $m \neq k$, $|y_{i(k),i(m)}| = 0$, then it is clear from equation (29) that $g_{1_o 2_o} \rightarrow \infty$. In such a case the resulting network will be an $(n+1)$-node realization of **Y**.

If $p = n$, then condition (19) does not arise at all and hence any dominant matrix **Y** with all off-diagonal entries negative is realizable by a $2n$-node network. This, in fact, is a result already well known.[9]

If $p = 2$, again condition (19) does not arise and the results of this section will then reduce to those of Reference 2.

Te main result of this section is summarized in the following theorem.

Theorem 1

A real symmetric dominant matrix **Y** with all off-diagonal entries negative is realizable as the s.c. conductance matrix of an $(n+p)$-node, $p \geq 1$, resistive n-port network containing no negative conductances provided that the entries of **Y** satisfy the following:

$$\frac{\Delta_{i(k),j}}{\Delta_{i(k),m}} = \frac{\Delta_{i(r),j}}{\Delta_{i(r),m}}, \qquad \begin{array}{l} i, j, m = 1, 2 \ldots p \\ i \neq j \neq m, \\ k, r = 1, 2 \ldots n_i \\ k \neq r \end{array}$$

We shall now illqstrate these discussions with an example.

Example

Consider the matrix **Y** given by

$$\mathbf{Y} = \left[\begin{array}{c|cc|c|c} & P_{1(1)} & P_{1(2)} & P_{2(1)} & P_{3(1)} \\ P_{1(1)} & 8 & -2 & -3 & -2 \\ P_{1(2)} & -2 & 14 & -6 & -4 \\ \hline P_{2(1)} & -3 & -6 & 13 & -3 \\ \hline P_3(1) & -2 & -4 & -3 & 12 \end{array}\right]$$

Let this be required to be realized as the s.c. conductance matrix of a 4-port resistive network having the port-configuration of Figure 2. The rows and columns of **Y** are designated suitably as shown above.

We then obtain the following

$$\Delta_{1(1),2} = 3 \qquad \Delta_{2(1),1} = 9$$

$$\Delta_{1(1),3} = 2 \qquad \Delta_{2(1),3} = 3$$

$$\Delta_{1(2),2} = 6 \qquad \Delta_{3(1),1} = 6$$

$$\Delta_{1(2),3} = 4 \qquad \Delta_{3(1),2} = 3$$

$$\Delta_{12} = 9 \qquad \Delta_{13} = 6 \qquad \Delta_{23} = 3$$

It can be seen from the above values of $\Delta_{i(k),j}$'s that the matrix **Y** satisfies the conditions of equation (19).

α_{ij}'s are then obtained using the definition given in equation (21).

$$\alpha_{12} = 1$$

$$\alpha_{13} = \tfrac{2}{3}$$

$$\alpha_{23} = \tfrac{1}{3}$$

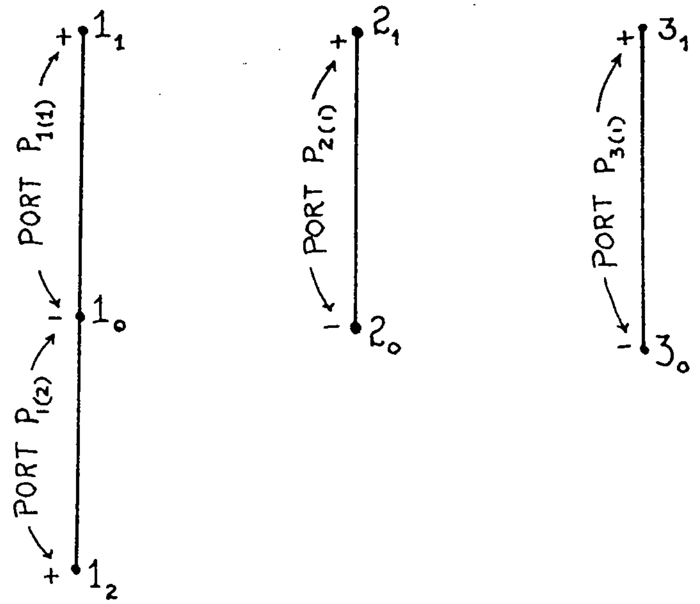

Figure 2. Port-configuration for the Example

Inequality (29) gives rise to the following:

$$|y_{1(1),1(2)}| = 2 \geqslant \frac{\Delta_{1(1),2}\Delta_{1(2),2}\alpha_{12}^2}{\alpha_{12}g_{1_02_0}-\Delta_{12}} + \frac{\Delta_{1(1),3}\Delta_{1(2),3}\alpha_{13}^2}{\alpha_{13}g_{1_02_0}-\Delta_{13}}$$

$$= \frac{3\times 6}{g_{1_02_0}-9} + \frac{2\times 4(\frac{4}{9})}{(\frac{2}{3})g_{1_02_0}-6}$$

$$= \frac{18}{g_{1_02_0}-9} + \frac{(\frac{32}{9})}{(\frac{2}{3})g_{1_02_0}-6}$$

A choice of 24 for $g_{1_02_0}$ will satisfy the above inequality. With this choice we obtain the following

$$g_{1_03_0} = 16; \qquad S_{12} = \tfrac{192}{5}; \qquad S_{23} = \tfrac{64}{5}$$

$$g_{2_03_0} = 8; \qquad S_{13} = \tfrac{128}{5}$$

$$k_{1(1),2} = k_{1(1),3} = \tfrac{1}{8}$$

$$k_{1(2),2} = k_{1(2),3} = \tfrac{1}{4}$$

$$k_{2(1),1} = k_{2(1),3} = \tfrac{3}{8}$$

$$k_{3(1),1} = k_{3(1),2} = \tfrac{3}{8}$$

The network of departure N_d (using equations (7), (8), (9) and (10)) is obtained as in Figure 3. The padding network is obtained (using equations (11), (12) and (13)) as in Figure (4).

The parallel combination of N_d and N_p is shown in Figure 5. This network realizes the matrix **Y**.

3. CONCLUSION

In this paper we have established the conditions sufficient for the $(n+p)$-node realizability of **Y**-matrices of $(n+1)$-node resistive n-port networks. The realization procedure given in Section 2 generalizes the ones known already for the cases of $p=n$ and $p=2$.[2,5,6,9] In fact, the conditions implied by equation (19)

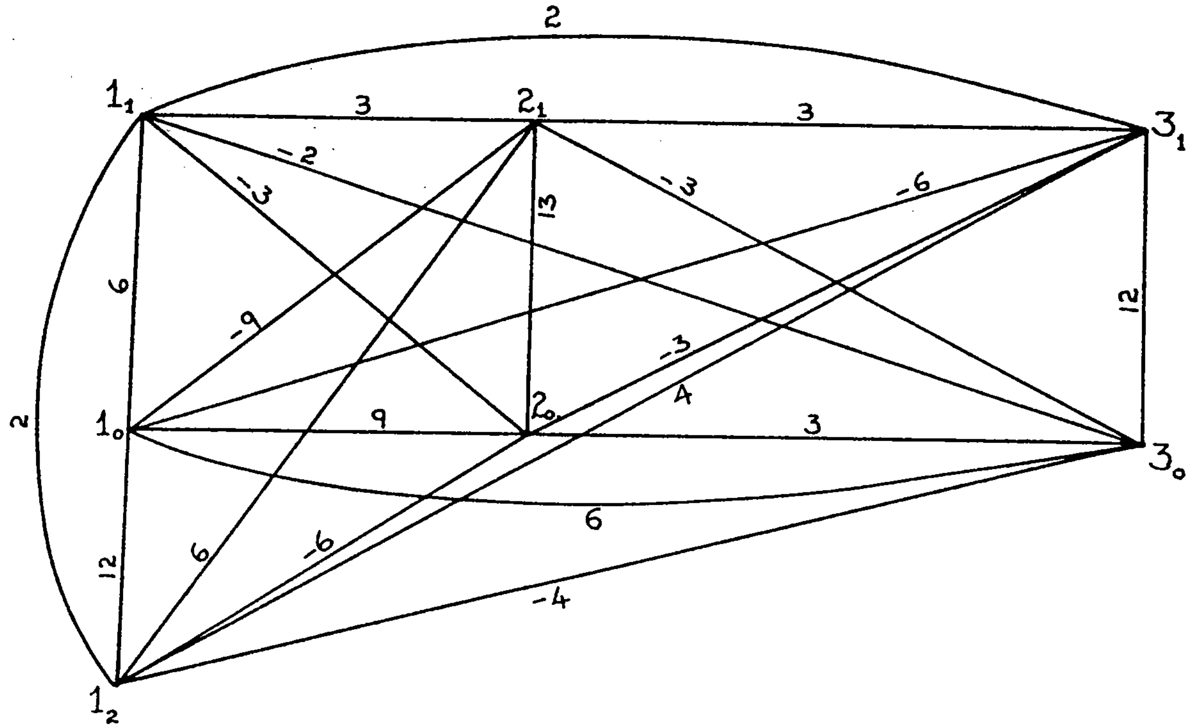

Figure 3. Network of departure for the Example

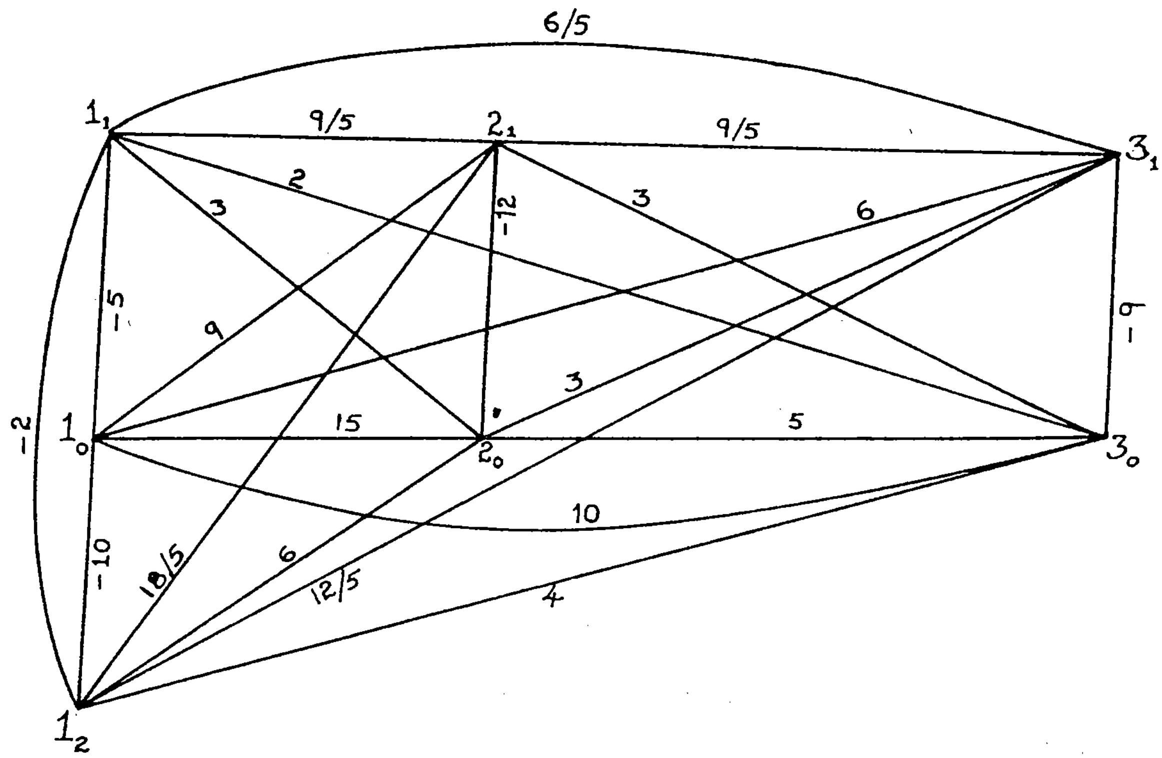

Figure 4. Padding network for the Example