# 7

# Graph Theory




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## 7.1 Introduction

Graph theory had its beginning in Euler's solution of what is known as the Konigsberg Bridge problem. Kirchhoff developed the theory of trees in 1847 as a tool in the study of electrical networks. This was the first application of graph theory to a problem in physical science. Electrical network theorists have since played a major role in the phenomenal advances of graph theory that have taken place in this century. A comprehensive treatment of these developments may be found in [1]. In this chapter we develop most of those results which form the foundation of graph theoretic study of electrical networks.

Our development of graph theory is self-contained, except for the definitions of standard set-theoretic operations and elementary results from matrix theory. We wish to note that the **ring sum** of two sets $S_1$ and $S_2$ refers to the set consisting of all those elements which are in $S_1$ or in $S_2$ but not in $S_1$ and $S_2$.

## 7.2 Basic Concepts

A graph $G = (V, E)$ consists of two sets: a finite set $V = (v_1, v_2, \ldots, v_n)$ of elements called **vertices** and a finite set $E = (e_1, e_2, \ldots, e_m)$ of elements called **edges**. Each edge is identified with a pair of vertices. If the edges of $G$ are identified with ordered pairs of vertices, then $G$ is called a **directed** or an **oriented graph**. Otherwise $G$ is called an **undirected** or a **nonoriented graph**. Graphs are amenable for pictorial representations. In a pictorial representation each vertex is represented by a dot and each edge is represented by a line segment joining the dots associated with the edge. In directed graphs we assign an orientation or direction to each edge. If the edge


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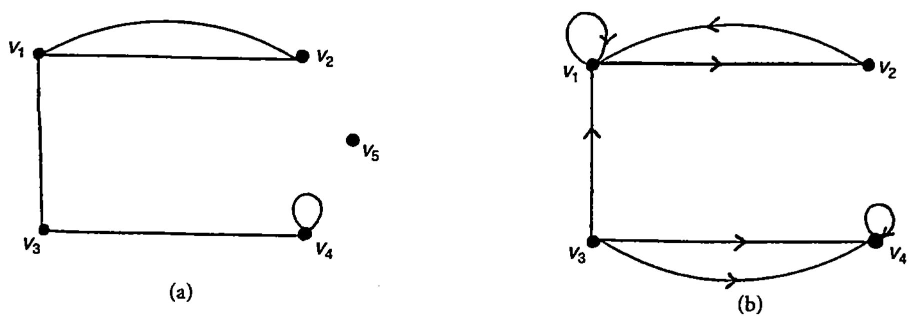
is associated with the ordered pair $(v_i, v_j)$, then this edge is oriented from $v_i$ to $v_j$. If an edge $e$ connects vertices $v_i$ and $v_j$ then it is denoted by $e = (v_i, v_j)$. In a directed graph $(v_i, v_j)$ refers to an edge directed from $v_i$ to $v_j$. A graph and a directed graph are shown in Fig. 7.1. Unless explicitly stated, the term "graph" may refer to an undirected graph or to a directed graph.




7.1 (a) An undirected graph; (b) a directed graph.

The vertices $v_i$ and $v_j$ associated with an edge are called the **end vertices** of the edge. All edges having the same pair of end vertices are called **parallel edges**. In a directed graph parallel edges refer to edges connecting the same pair of vertices $v_i$ and $v_j$ and oriented in the same way from $v_i$ to $v_j$ or from $v_j$ to $v_i$. For instance, in the graph of Fig. 7.1(a), the edges connecting $v_1$ and $v_2$ are parallel edges. In the directed graph of Fig. 7.1(b) the edges connecting $v_3$ and $v_4$ are parallel edges. However, the edges connecting $v_1$ and $v_2$ are not parallel edges because they are not oriented the same way. If the end vertices of an edge are not distinct, then the edge is called a **self-loop**. The graph of Fig. 7.1(a) has one self-loop and the graph of Fig. 7.1(b) has two self-loops.

An edge is said to be **incident on** its end vertices. In a directed graph the edge $(v_i,v_j)$ is said to be **incident out** of $v_i$ and is said to be **incident into** $v_j$. Vertices $v_i$ and $v_j$ are adjacent if an edge connects $v_i$ and $v_j$.

The number of edges incident on a vertex $v_i$ is called the **degree** of $v_i$ and is denoted by $d(v_i)$. In a directed graph $d_{\text{in}}(v_i)$ refers to the number of edges incident into vertex $v_i$, and it is called the **in-degree** of $v_i$. $d_{\text{out}}(v_i)$ refers to the number of edges incident out of vertex $v_i$ and it is called the **out-degree** of $v_i$. If $d(v_i) = 0$, then $v_i$ is called an **isolated vertex**. If $d(v_i) = 1$, then $v_i$ is called a **pendant vertex**. A self-loop at a vertex $v_i$ is counted twice while computing $d(v_i)$. As an example, in the graph of Fig. 7.1(a), $d(v_1) = 3$, $d(v_4) = 3$, and $v_5$ is an isolated vertex. In the directed graph of Fig. 7.1(b) $d_{\text{in}}(v_1) = 3$, $d_{\text{out}}(v_1) = 2$.

Note that in a directed graph, for every vertex $v_i$,

$$d(v_i) = d_{\text{in}}(v_i) + d_{\text{out}}(v_i)$$

*THEOREM 7.1*

1. *The sum of the degrees of the vertices of a graph G is equal to 2m, where m is the number of edges of G.*
2. *In a directed graph with m edges, the sum of the in-degrees and the sum of the out-degrees are both equal to m.*

**PROOF 7.1**

1. Because each edge is incident on two vertices, it contributes 2 to the sum of the degrees

of $G$. Hence, all edges together contribute $2m$ to the sum of the degrees.

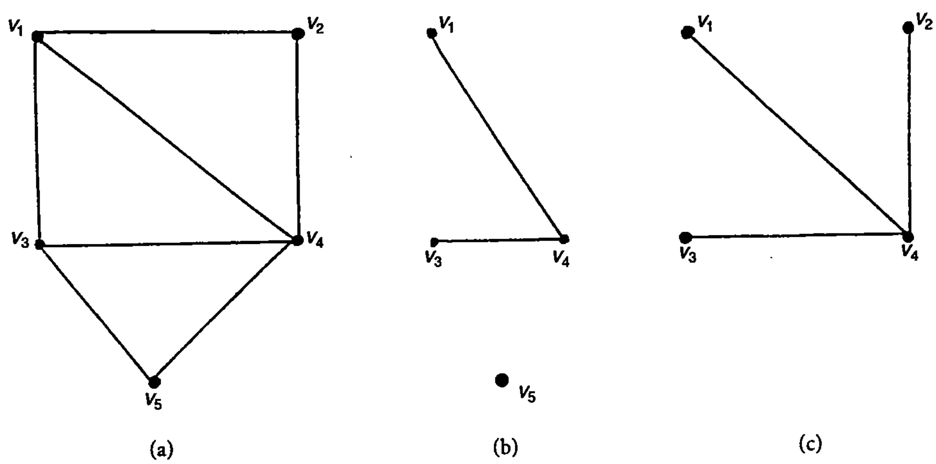
2. Proof follows if we note that each edge is incident out of exactly one vertex and incident into exactly one vertex.

□

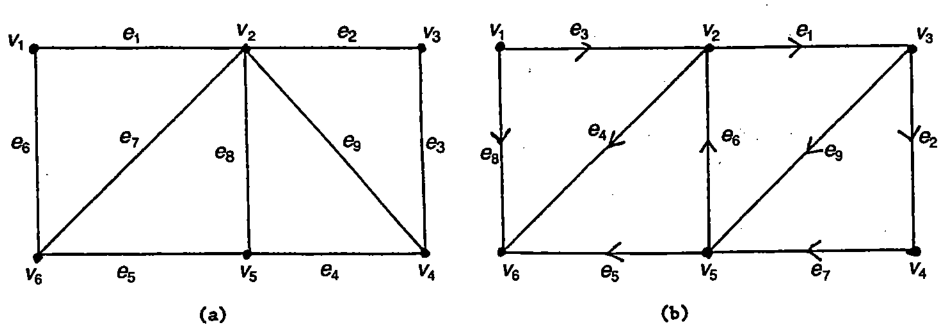
*THEOREM 7.2 The number of vertices of odd degree in any graph is even.*

**PROOF 7.2** By Theorem 7.1, the sum of the degrees of the vertices is even. Thus, the sum of the odd degrees must be even. This is possible only if the number of vertices of odd degree is even. □

Consider a graph $G = (V, E)$. The graph $G' = (V', E')$ is a subgraph of $G$ if $V' \subseteq V$ and $E' \subseteq E$. If every vertex in $V'$ is an end vertex of an edge in $E'$, then $G'$ is called the **induced subgraph** of $G$ on $E'$. As an example, a graph $G$ and two subgraphs of $G$ are shown in Fig. 7.2.




7.2 (a) Graph $G$; (b) subgraph of $G$; (c) an edge-induced subgraph of $G$.



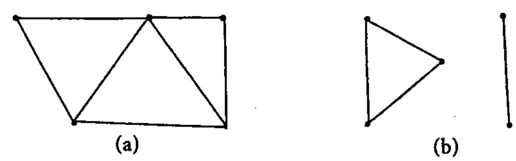

7.3 (a) An undirected graph; (b) a directed graph.

In a graph $G$ a **path** $P$ connecting vertices $v_i$ and $v_j$ is an alternating sequence of vertices and edges starting at $v_i$ and ending at $v_j$, with all vertices except $v_i$ and $v_j$ being distinct. In a directed graph a *path* $P$ connecting vertices $v_i$ and $v_j$ is called a **directed path** from $v_i$ to $v_j$ if all the edges in $P$ are oriented in the same direction as we traverse $P$ from $v_i$ toward $v_j$. If a path starts and ends at the same vertex, it is called a

**circuit.**[1] In a directed graph, a circuit in which all the edges are oriented in the same direction is called a **directed circuit**. It is often convenient to represent paths and circuits by the sequences of edges representing them.

For example, in the undirected graph of Fig. 7.3(a) $P$: $e_1, e_2, e_3, e_4$ is a path connecting $v_1$ and $v_5$ and $C$: $e_1, e_2, e_3, e_4, e_5, e_6$ is a circuit. In the directed graph of Fig. 7.3(b) $P : e_1, e_2, e_7, e_5$ is a directed path and $C : e_1, e_2, e_7, e_6$ is a directed circuit. Note that $e_7, e_5, e_4, e_1, e_2$ is a circuit in this directed graph, although it is not a directed circuit.

Two vertices $v_i$ and $v_j$ are said to be **connected** in a graph $G$ if a path in $G$ connects $v_i$ and $v_j$. A graph $G$ is **connected** if every pair of vertices in $G$ is connected; otherwise it is a **disconnected graph**. For example, the graph $G$ in Fig. 7.4(a) is connected, but the graph in Fig. 7.4(b) is not connected.




7.4 (a) A connected graph; (b) a disconnected graph.

A connected subgraph $G' = (V', E')$ of a graph $G = (V, E)$ is a **component** of $G$ if adding to $G'$ an edge $e \in E - E'$ results in a disconnected graph. Thus, a connected graph has exactly one component. For example, the graph in Fig. 7.4(b) is not connected and has two components.

A **tree** is a graph that is connected and has no circuits. Consider a connected graph $G$. A subgraph of $G$ is a **spanning tree**[2] of $G$ if the subgraph is a tree and contains all the vertices of $G$. A tree and a spanning tree of the graph of Fig. 7.5(a) are shown in Fig. 7.5(b) and (c), respectively.

The edges of a spanning tree $T$ are called the **branches** of $T$. Given a spanning tree of a connected graph G, the **cospanning tree**[2] relative to $T$ is the subgraph of $G$ induced by the edges that are not present in $T$. For example, the cospanning tree relative to the spanning tree $T$ of Fig. 7.5(c) consists of the edges $e_3$, $e_6$, and $e_7$. The edges of a cospanning tree are called **chords.**

A subgraph of a graph $G$ is a **k-tree** of $G$ if the subgraph has exactly $k$ components and has no circuits. For example, a 2-tree of the graph of Fig. 7.5(a) is shown in Fig. 7.5(d). If a graph has $k$ components, then a **forest** of $G$ is a spanning subgraph that has $k$ components and no circuits. Thus, each component of the forest is a spanning tree of a component of $G$. A graph $G$ and a forest of $G$ are shown in Fig. 7.6.

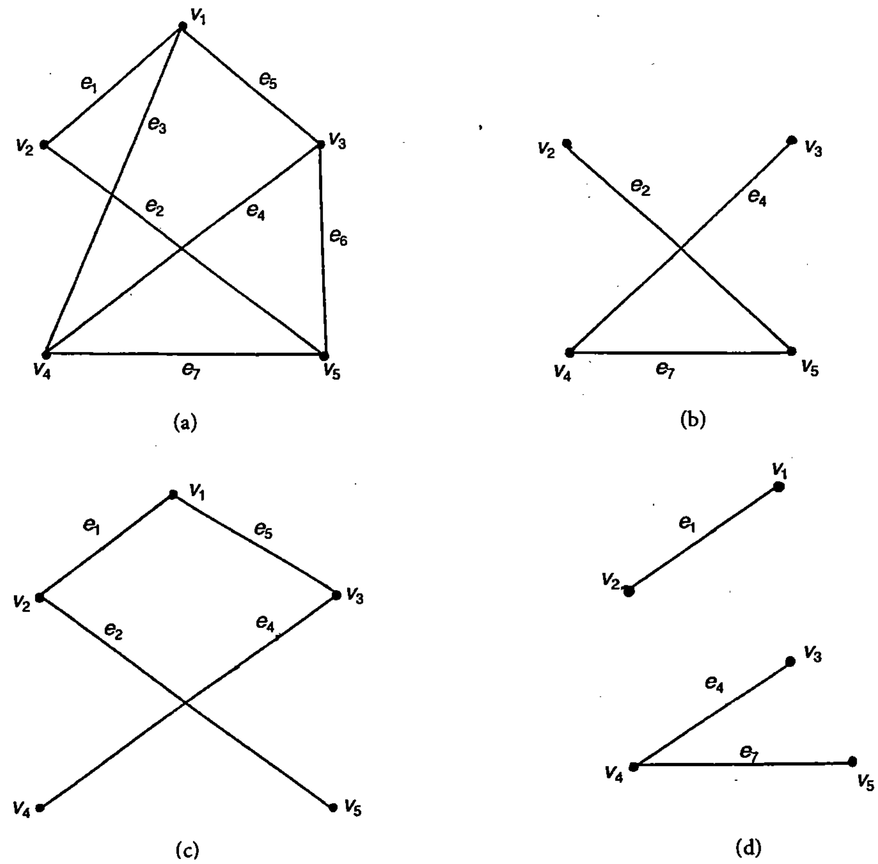
Consider a directed graph $G$. A spanning tree $T$ of $G$ is called a **directed spanning tree** with root $v_i$ if $T$ is a spanning tree of $G$, and $d_{\text{in}}(v_i) = 0$ and $d_{\text{in}}(v_j) = 1$ for all $v_j \neq v_i$. A directed graph $G$ and a directed spanning tree with root $v_1$ are shown in Fig. 7.7.

It can easily be verified that in a tree exactly one path connects any two vertices.

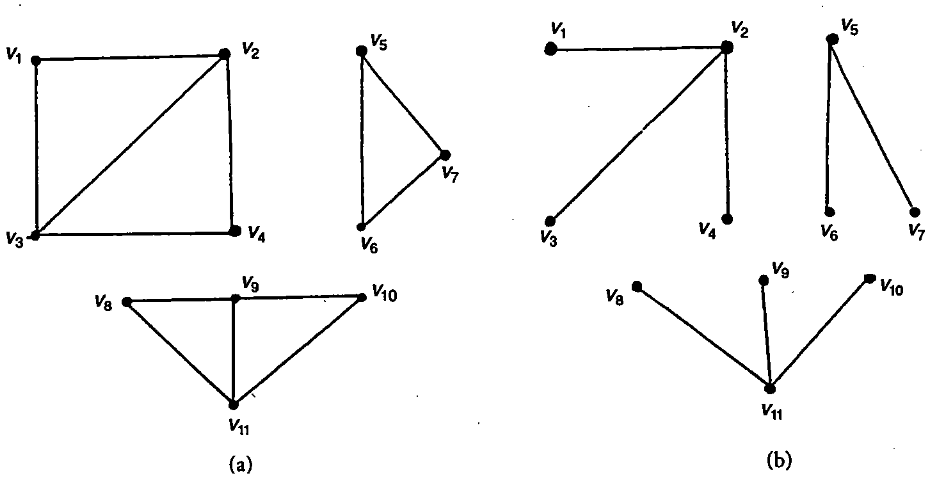
*THEOREM 7.3* *A tree on n vertices has n − 1 edges.*

[1] In electrical network theory literature the term **loop** is also used to refer to a circuit.

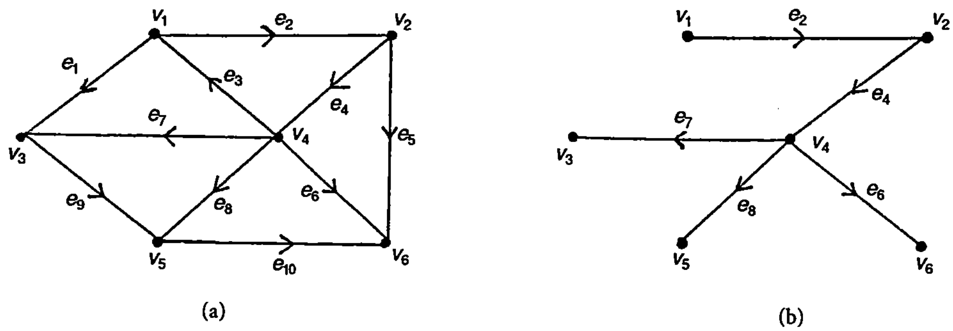
[2] In electrical network theory literature the terms **tree** and **cotree** are usually used to mean spanning tree and cospanning tree, respectively.




**7.5** (a) Graph $G$; (b) a tree of graph $G$; (c) a spanning tree of $G$; (d) a 2-tree of $G$.




**7.6** (a) Graph $G$; (b) a forest of $G$.




7.7 (a) Directed graph $G$; (b) a directed spanning tree of $G$ with root $v_1$.

**PROOF 7.3** Proof is by induction on the number of vertices of the tree. Clearly, the result is true if a tree has one or two vertices. Assume that the result is true for trees on $n \geq 2$ or fewer vertices. Consider now a tree $T$ on $n+1$ vertices. Pick an edge $e = (v_i, v_j)$ in $T$. Removing $e$ from $T$ would disconnect it into exactly two components $T_1$ and $T_2$. Both $T_1$ and $T_2$ are themselves trees. Let $n_1$ and $m_1$ be the number of vertices and the number of edges in $T_1$, respectively. Similarly $n_2$ and $m_2$ are defined. Then, by the induction hypothesis

$$m_1 = n_1 - 1$$

and

$$m_2 = n_2 - 1$$

Thus, the number $m$ of edges in $T$ is given by

$$\begin{aligned} m &= m_1 + m_2 + 1 \\ &= (n_1 - 1) + (n_2 - 1) + 1 \\ &= n_1 + n_2 - 1 \\ &= n - 1 \end{aligned}$$

This completes the proof of the theorem. □

If a connected graph $G$ has $n$ vertices, $m$ edges, and $k$ components, then the **rank** $\rho$ and **nullity** $\mu$ of $G$ are defined as follows:

$$\rho(G) = n - k \tag{7.1}$$
$$\mu(G) = m - n + k \tag{7.2}$$

Clearly, if $G$ is connected, then any spanning tree of $G$ has $\rho = n - 1$ branches and $\mu = m - n + 1$ chords.

We conclude this subsection with the following theorems. Proofs of these theorems may be found in [2].

*THEOREM 7.4* *A tree on $n \geq 2$ vertices has at least two pendant vertices.*

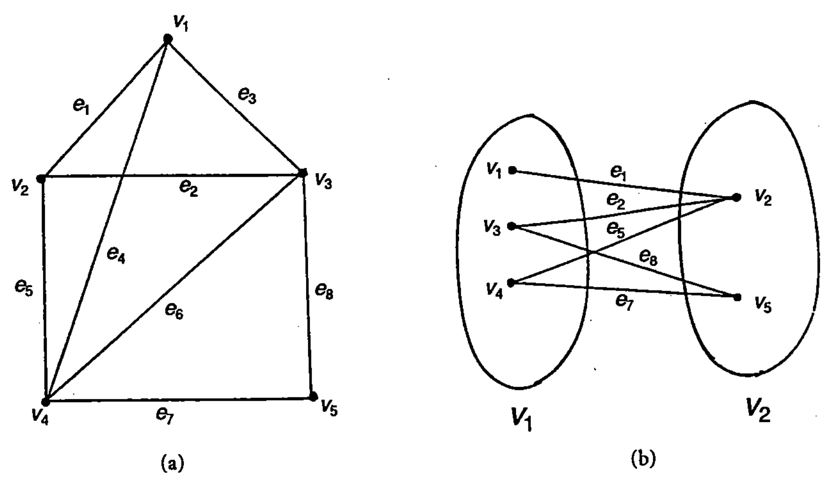
*THEOREM 7.5* *A subgraph of an $n$-vertex connected graph $G$ is a spanning tree of $G$ if and only if the subgraph has no circuits and has $n-1$ edges.*

***THEOREM 7.6*** *If a subgraph $G'$ of a connected graph $G$ has no circuits then there exists a spanning tree of $G$ that contains $G'$.*

## 7.3 Cuts, Circuits, and Orthogonality

We introduce here the notions of a cut and a cutset and develop certain results which bring out the dual nature of circuits and cutsets.

Consider a connected graph $G = (V, E)$ with $n$ vertices and $m$ edges. Let $V_1$ and $V_2$ be two mutually disjoint nonempty subsets of $V$ such that $V = V_1 \cup V_2$. Thus, $V_2 = \overline{V_1}$, the complement of $V_1$ in $V$. $V_1$ and $V_2$ are also said to form a partition of $V$. Then the set of all those edges which have one end vertex in $V_1$ and the other in $V_2$ is called a **cut** of $G$ and is denoted by $< V_1, V_2 >$. As an example, a graph $G$ and a cut $< V_1, V_2 >$ of $G$ are shown in Fig. 7.8.




**7.8** (a) Graph $G$; (b) cut $\langle v_1, v_2 \rangle$ of $G$.

The graph $G'$ which results after removing the edges in a cut will have at least two components and so will not be connected. $G'$ may have more than two components. A **cutset** $S$ of a connected graph $G$ is a minimal set of edges of $G$ such that removal of $S$ disconnects $G$ into exactly two components. Thus, a cutset is also a cut. Note that the minimality property of a cutset implies that no proper subset of a cutset is a cutset.

Consider a spanning tree $T$ of a connected graph $G$. Let $b$ be a branch of $T$. Removal of the branch $b$ disconnects $T$ into exactly two components, $T_1$ and $T_2$. Let $V_1$ and $V_2$ denote the vertex sets of $T_1$ and $T_2$, respectively. Note that $V_1$ and $V_2$ together contain all the vertices of $G$. We can verify that the cut $< V_1, V_2 >$ is a cutset of $G$ and is called the **fundamental cutset** of $G$ with respect to branch $b$ of $T$. Thus, for a given connected graph $G$ and a spanning tree $T$ of $G$, we can construct $n - 1$ fundamental cutsets, one for each branch of $T$. As an example, for the graph shown in Fig. 7.8, the fundamental cutsets with respect to the spanning tree $T = [e_1, e_2, e_6, e_8]$ are

| | |
|---|---|
| Branch $e_1$ : | $(e_1, e_3, e_4)$ |
| Branch $e_2$ : | $(e_2, e_3, e_4, e_5)$ |
| Branch $e_6$ : | $(e_6, e_4, e_5, e_7)$ |
| Branch $e_8$ : | $(e_8, e_7)$ |

Note that the fundamental cutset with respect to branch $b$ contains $b$. Furthermore, the branch $b$ is not present in any other fundamental cutset with respect to $T$.

Next we identify a special class of circuits of a connected graph $G$. Again, let $T$ be a spanning tree of $G$. Because exactly one path exists between any two vertices of $T$, adding a chord $c$ to $T$ produces a unique circuit. This circuit is called the **fundamental circuit** of $G$ with respect to chord $c$ of $T$. Note again that the fundamental circuit with respect to chord $c$ contains $c$, and the chord $c$ is not present in any other fundamental circuit with respect to $T$. As an example, the set of fundamental circuits with respect to the spanning tree $T = (e_1, e_2, e_6, e_8)$ of the graph shown in Fig. 7.8 is

| | |
|---|---|
| Chord $e_3$ : | $(e_3, e_1, e_2)$ |
| Chord $e_4$ : | $(e_4, e_1, e_2, e_6)$ |
| Chord $e_5$ : | $(e_5, e_2, e_6)$ |
| Chord $e_7$ : | $(e_7, e_8, e_6)$ |

We now present a result which is the basis of what is known as the orthogonality relationship.

*THEOREM 7.7* *A circuit and a cutset of a connected graph have an even number of common edges.*

**PROOF 7.4** Consider a circuit $C$ and a cutset $S = \langle V_1, V_2 \rangle$ of $G$. The result is true if $C$ and $S$ have no common edges. Suppose that $C$ and $S$ possess some common edges. Let us traverse the circuit $C$ starting from a vertex, e.g., $v_1$ in $V_1$. Because the traversing should end at $v_1$, it is necessary that every time we encounter an edge of $S$ leading us from $V_1$ to $V_2$ an edge of $S$ must lead from $V_2$ back to $V_1$. This is possible only if $S$ and $C$ have an even number of common edges. □

The above result is the foundation of the theory of duality in graphs. Several applications of this simple result are explored in different parts of this chapter.

A comprehensive treatment of the duality theory and its relationship to planarity may be found in [2]. The following theorem establishes a close relationship between fundamental circuits and fundamental cutsets.

*THEOREM 7.8*

1. *The fundamental circuit with respect to a chord of a spanning tree $T$ of a connected graph consists of exactly those branches of $T$ whose fundamental cutsets contain the chord.*
2. *The fundamental cutset with respect to a branch of a spanning tree $T$ of a connected graph consists of exactly those chords of $T$ whose fundamental circuits contain the branch.*

**PROOF 7.5** Let $C$ be the fundamental circuit of a connected graph $G$ with respect to a chord $c$ of a spanning tree $T$ of $G$. Let $C$ contain, in addition to the chord $c$, the branches $b_1, b_2, \ldots, b_k$ of $T$. Let $S_i$ be the fundamental cutset with respect to branch $b_i$.

We first show that each $S_i$, $1 \leq i \leq k$ contains $c$. Note that $b_i$ is the only branch common to $S_i$ and $C$, and $c$ is the only chord in $C$. Because by Theorem 7.7, $S_i$ and $C$ must have an even number of common edges, it is necessary that $S_i$ contains $c$.

Next we show that no other fundamental cutset of $T$ contains $c$. Suppose the fundamental cutset $S_{k+1}$ with respect to some branch $b_{k+1}$ of $T$ contains $c$. Then $c$ will be the only edge common to $S_{k+1}$ and $C$, contradicting Theorem 7.7. Thus the chord $c$ is present only in those cutsets defined by the branches $b_1, b_2, \ldots, b_k$.

The proof for item 2 of the theorem is similar to that of item 1. □

## 7.4 Incidence, Circuit, and Cut Matrices of a Graph

The incidence, circuit, and cut matrices are coefficient matrices of Kirchhoff's equations which describe an electrical network. We develop several properties of these matrices which have proved useful in the study of electrical networks. Our discussions are mainly in the context of directed graphs. The results become valid in the case of undirected graphs if addition and multiplication are in $GF(2)$, the field of integers modulo 2. (Note that $1 + 1 = 0$ in this field.)

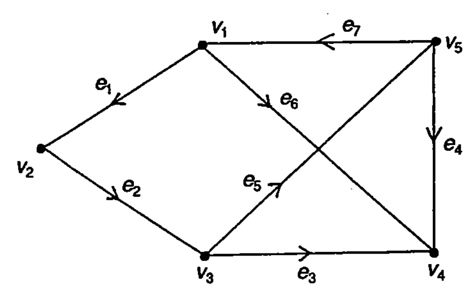
### Incidence Matrix

Consider a connected directed graph $G$ with $n$ vertices and $m$ edges and having no self-loops. The **all-vertex incidence matrix** $A_c = [a_{ij}]$ of $G$ has $n$ rows, one for each vertex, and $m$ columns, one for each edge. The element $a_{ij}$ of $A_c$ is defined as follows:

$$a_{ij} = \begin{cases} 1, & \text{if the } j\text{th edge is incident out of the } i\text{th vertex,} \\ -1, & \text{if the } j\text{th edge is incident into the } i\text{th vertex,} \\ 0, & \text{if the } j\text{th edge is not incident on the } i\text{th vertex} \end{cases}$$

A row of $A_c$ will be referred to as an **incidence vector**. As an example, for the directed graph shown in Fig. 7.9, the matrix $A_c$ is given below.

$$A_c = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \end{array}$$




**7.9** A directed graph.

From the definition of $A_c$ it should be clear that each column of this matrix has exactly two nonzero entries, one $+1$ and one $-1$, and therefore, we can obtain any row of $A_c$ from the remaining rows. Thus,

$$\text{rank}(A_c) \leq n - 1 \tag{7.3}$$

An $(n - 1)$ rowed submatrix of $A_c$ is referred to as an **incidence matrix** of $G$. The vertex which corresponds to the row of $A_c$ that is not in $A$ is called the **reference vertex** of $A$.

***THEOREM 7.9*** *The determinant of an incidence matrix of a tree is* $\pm 1$.

**PROOF 7.6** Proof is by induction on the number $m$ of edges in the tree. We can easily verify that the result is true for any tree with $m \leq 2$ edges. Assume that the result is true for all trees having $m \geq 2$ or fewer edges. Consider a tree $T$ with $m+1$ edges. Let $A$ be the incidence matrix of $T$ with reference vertex $v_r$. Because, by Theorem 7.4, $T$ has at least two pendant vertices, we can find a pendant vertex $v_i \neq v_r$. Let $(v_i, v_j)$ be the only edge incident on $v_i$. Then the remaining edges form a tree $T_1$. Let $A_1$ be the incidence matrix of $T_1$ with vertex $v_r$ as reference. Now let us rearrange the rows and columns of $A$ so that the first $n-1$ rows correspond to the vertices in $T_1$, (except $v_r$) and the first $n-1$ columns correspond to the edges of $T_1$. Then we have

$$A = \begin{bmatrix} A_1 & A_3 \\ 0 & \pm 1 \end{bmatrix}$$

So

$$\det A = \pm(\det A_1) \tag{7.4}$$

$A_1$ is the incidence matrix of $T_1$ and $T_1$ has $m$ edges, it follows from the induction hypothesis that $\det A_1 = \pm 1$. Hence the theorem. □

Because a connected graph has at least one spanning tree, it follows from the above theorem that any incidence matrix $A$ of a connected graph has a nonsingular submatrix of order $n-1$. Therefore,

$$\operatorname{rank}(A_c) \geq n-1 \tag{7.5}$$

Combining (7.3) and (7.5) yields the following theorem.

***THEOREM 7.10*** *The rank of any incidence matrix of a connected directed graph G is equal to $n-1$, the rank of G.*

## Cut Matrix

Consider a cut $\langle V_a, \overline{V_a} \rangle$ in a connected directed graph $G$ with $n$ vertices and $m$ edges. Recall that $\langle V_a, \overline{V_a} \rangle$ consists of all those edges connecting vertices in $V_a$ to those in $\overline{V_a}$. This cut may be assigned an orientation from $V_a$ to $\overline{V_a}$ or from $\overline{V_a}$ to $V_a$. Suppose the orientation of $\langle V_a, \overline{V_a} \rangle$ is from $V_a$ to $\overline{V_a}$. Then the orientation of an edge $(v_i, v_j)$ is said to agree with the cut orientation if $v_i \in V_a$, and $v_j \in \overline{V_a}$.

The **cut matrix** $Q_c = [q_{ij}]$ of $G$ has $m$ columns, one for each edge, and has one row for each cut. The element $q_{ij}$ is defined as follows:

$$q_{ij} = \begin{cases} 1, & \text{if the } j\text{th edge is in the } i\text{th cut and its orientation agrees with} \\ & \text{the cut orientation,} \\ -1, & \text{if the } j\text{th edge is in the } i\text{th cut and its orientation does not} \\ & \text{agree with the cut orientation,} \\ 0, & \text{if the } j\text{th edge is not in the } i\text{th cut} \end{cases}$$

Each row of $Q_c$ is called a **cut vector**.

The edges incident on a vertex form a cut. Thus it follows that the matrix $A_c$ is a submatrix of $Q_c$. Next we identify another important submatrix of $Q_c$.

Recall that each branch of a spanning tree $T$ of a connected graph $G$ defines a fundamental cutset. The submatrix of $Q_c$ corresponding to the $n-1$ fundamental cutsets defined by $T$ is called the **fundamental cutset matrix** $Q_f$ of $G$ with respect to $T$.

Let $b_1, b_2, \ldots, b_{n-1}$ denote the branches of $T$. Let us assume that the orientation of a fundamental cutset is chosen so as to agree with that of the defining branch. Suppose we arrange the rows and the columns of $Q_f$ so that the $i$th column corresponds to branch $b_i$, and the $i$th row corresponds to the fundamental cutset defined by $b_i$. Then the matrix $Q_f$ can be displayed in a convenient form as follows:

$$Q_f = \left[U \mid Q_{fc}\right] \tag{7.6}$$

where $U$ is the unit matrix of order $n-1$ and its columns correspond to the branches of $T$.

As an example, the fundamental cutset matrix of the graph in Fig. 7.9 with respect to the spanning tree $T = (e_1, e_2, e_5, e_6)$ is given below:

$$Q_f = \begin{array}{c} \\ e_1 \\ e_2 \\ e_5 \\ e_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_5 & e_6 & e_3 & e_4 & e_7 \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \end{array}$$

It is clear from (7.6) that the rank of $Q_f$ is $n-1$. Hence,

$$\text{rank}(Q_c) \geq n-1 \tag{7.7}$$

## Circuit Matrix

Consider a circuit $C$ in a connected directed graph $G$ with $n$ vertices and $m$ edges. This circuit can be traversed in one of two directions, clockwise or counterclockwise. The direction we choose for traversing $C$ is called the orientation of $C$. If an edge $e = (v_i, v_j)$ directed from $v_i$ to $v_j$ is in $C$, and if $v_i$ appears before $v_j$ as we traverse $C$ in the direction specified by its orientation, then we say that the orientation of $e$ agrees with the orientation of $C$.

The **circuit matrix** $B_c = [b_{ij}]$ of $G$ has $m$ columns, one for each edge, and has one row for each circuit in $G$. The element $b_{ij}$ is defined as follows:

$$b_{ij} = \begin{cases} 1, & \text{if the } j\text{th edge is in the } i\text{th circuit, and its orientation agrees} \\ & \text{with the circuit orientation,} \\ -1, & \text{if the } j\text{th edge is in the } i\text{th circuit, and its orientation does not} \\ & \text{agree with the circuit orientation,} \\ 0, & \text{if the } j\text{th edge is not in the } i\text{th circuit} \end{cases}$$

Each row of $B_c$ is called a **circuit vector.**

The submatrix of $B_c$ corresponding to the fundamental circuits defined by the chords of a spanning tree $T$ is called the **fundamental circuit matrix** $B_f$ of $G$ with respect to the spanning tree $T$.

Let $c_1, c_2, \ldots, c_{m-n+1}$ denote the chords of $T$. Suppose we arrange the columns and the rows of $B_f$ so that the $i$th row corresponds to the fundamental circuit defined by the chord $c_i$, and the $i$th column corresponds to the chord $c_i$.

If, in addition, we choose the orientation of a fundamental circuit to agree with the orientation of the defining chord, we can write $B_f$ as

$$B_f = \left[U | B_{ft}\right] \tag{7.8}$$

where $U$ is the unit matrix of order $m-n+1$, and its columns correspond to the chords of $T$.

As an example, the fundamental circuit matrix of the graph shown in Fig. 7.9 with respect to the tree $T = (e_1, e_2, e_5, e_6)$ is given below:

$$B_f = \begin{array}{c} \\ e_3 \\ e_4 \\ e_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_3 & e_4 & e_7 & e_1 & e_2 & e_5 & e_6 \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right] \end{array}$$

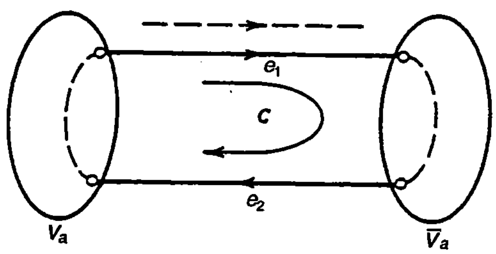
It is clear from (7.8) that the rank of $B_f$ is $m - n + 1$. Hence,

$$\text{rank}(B_c) \geq m - n + 1 \tag{7.9}$$

## 7.5 Orthogonality Relation and Ranks of Circuit and Cut Matrices

**THEOREM 7.11** *If a cut and a circuit in a directed graph have 2k edges in common, then k of these edges have the same relative orientation in the cut and in the circuit, and the remaining k edges have one orientation in the cut and the opposite orientation in the circuit.*

**PROOF 7.7** Consider a cut $\langle V_a, \overline{V_a}\rangle$ and a circuit $C$ in a directed graph. Suppose we traverse $C$ starting from a vertex in $V_a$. Then, for every edge $e_1$ that leads from $V_a$ to $\overline{V_a}$, an edge $e_2$ leads from $\overline{V_a}$ to $V_a$. Suppose the orientation of $e_1$ agrees with the orientation of the cut and that of the circuit. Then we can easily verify that $e_2$ has one orientation in the cut and the opposite orientation in the circuit (see Fig. 7.10). On the other hand, we can also verify that if $e_1$ has one orientation in the cut and the opposite orientation in the circuit, the $e_2$ will have the same relative orientation in the circuit and in the cut. This proves the theorem. □




**7.10** Relative orientations of an edge in a cut and a circuit.

Next we prove the orthogonality relation.

**THEOREM 7.12** *If the columns of the circuit matrix $B_c$ and the columns of the cut matrix $Q_c$ are arranged in the same edge order, then*

$$B_c Q_c^t = 0 \tag{7.10}$$

**PROOF 7.8** Each entry of the matrix $B_c Q_c^t$ is the inner product of a circuit vector and a cut vector. Suppose a circuit and a cut have $2k$ edges in common. The inner product of the corresponding vectors is zero, because by Theorem 7.11, this product is the sum of $k$ 1's and $k$ − 1's. □

The orthogonality relation is a profound result with interesting applications in electrical network theory. Consider a connected graph $G$ with $m$ edges and $n$ vertices. Let $Q_f$ be the fundamental cutset matrix and $B_f$ be the fundamental circuit matrix of $G$ with respect to a spanning tree $T$. If we write $Q_f$ and $B_f$ as in (7.6) and (7.8), then using the orthogonality relation we get

$$B_f Q_f^t = 0$$

that is,

$$[B_{ft}U]\begin{bmatrix} U \\ Q^t_{fc} \end{bmatrix} = 0$$

that is,

$$B_{ft} = -Q^t_{fc} \tag{7.11}$$

Using (7.11) each circuit vector can now be expressed as a linear combination of the fundamental circuit vectors. Consider a circuit vector $\beta = [\beta_1, \beta_2, \ldots, \beta_\rho | \beta_{\rho+1} \cdots \beta_m]$ of $G$ where $\rho = n - 1$, is the rank of $G$. Then, again by the orthogonality relation we have

$$\beta Q^t_f = [\beta_1, \beta_2, \ldots, \beta_\rho \,|\, \beta_{\rho+1} \cdots \beta_m]\begin{bmatrix} U \\ Q^t_{fc} \end{bmatrix} = 0 \tag{7.12}$$

Therefore,

$$\begin{aligned}[\beta_1, \beta_2, \ldots, \beta_\rho] &= -[\beta_{\rho+1}, \beta_{\rho+2} \cdots \beta_m]Q^t_{fc} \\ &= [\beta_{\rho+1}, \beta_{\rho+2} \cdots \beta_m]B_{ft}\end{aligned}$$

So,

$$\begin{aligned}[\beta_1, \beta_2, \ldots, \beta_m] &= [\beta_{\rho+1}, \beta_{\rho+2} \cdots \beta_m][B_{ft} \quad U] \\ &= [\beta_{\rho+1}, \beta_{\rho+2} \cdots \beta_m]B_f\end{aligned} \tag{7.13}$$

Thus, any circuit vector can be expressed as a linear combination of the fundamental circuit vectors. So

$$\text{rank}(B_c) \leq \text{rank}(B_f) = m - n + 1$$

Combining the above with (7.9) we obtain

$$\text{rank}(B_c) = m - n + 1 \tag{7.14}$$

Starting from a cut vector and using the orthogonality relation we can prove in an exactly similar manner that

$$\text{rank}(Q_c) \leq \text{rank}(Q_f) = n - 1$$

Combining the above with (7.7) we get

$$\text{rank}(Q_c) = n - 1$$

Summarizing, we have the following theorem.

***THEOREM 7.13*** *For a connected graph G with m edges and n vertices*

$$\begin{aligned}&\text{rank}(B_c) = m - n + 1 \\ &\text{rank}(Q_c) = n - 1\end{aligned}$$

We wish to note from (7.12) that the vector corresponding to a circuit $C$ can be expressed as an appropriate linear combination of the circuit vectors corresponding to the chords present in $C$. Similarly, the vector corresponding to a cut can be expressed as an appropriate linear combination of the cut vectors corresponding to the branches present in the cut. Because modulo 2 addition of two vectors corresponds to the ring sum of the corresponding subgraphs we have the following results for undirected graphs.

***THEOREM 7.14*** *Let G be a connected undirected graph.*

1. *Every circuit can be expressed as a ring sum of the fundamental circuits with respect to a spanning tree.*
2. *Every cut can be expressed as a ring sum of the fundamental cutsets with respect to a spanning tree.*

We can easily verify the following consequences of the orthogonality relation:

1. A linear relationship exists among the columns of the cut matrix (also of the incidence matrix) which correspond to the edges of a circuit.
2. A linear relationship exists among the columns of the circuit matrix which correspond to the edges of a cut.

The following theorem characterizes the submatrices of $A_c$, $Q_c$ and $B_c$ which correspond to spanning trees and cospanning trees. Proof follows from the above results and may be found in [2].

*THEOREM 7.15* *Let G be a connected graph G with n vertices, and m edges.*

1. *A square submatrix of order $n-1$ of $Q_c$ (also of $A_c$) is nonsingular iff the edges corresponding to the columns of this submatrix form a spanning tree of G.*
2. *A square submatrix of order $m-n+1$ of $B_c$ is nonsingular iff the edges corresponding to the columns of this submatrix form a cospanning tree of G.*

## 7.6 Spanning Tree Enumeration

Here we first establish a formula for counting the number of spanning tress of an undirected graph. We then state a generalization of this result for the case of a directed graph. These formulas have played key roles in the development of topological formulas for electrical network functions. A detailed development of topological formulas for network functions may be found in Swamy and Thulasiraman [1].

The formula for counting the number of spanning trees of a graph is based on Theorem 7.9 and a result in matrix theory, known as the Binet-Cauchy theorem.

A **major** of a matrix is a determinant of maximum order. Consider a matrix $P$ of order $p \times q$ and a matrix $Q$ of order $q \times p$, with $p \leq q$. The majors of $P$ and $Q$ are of order $p$. If a major of $P$ consists of columns $i_1, i_2, \ldots, i_p$ the corresponding major of $Q$ is formed by rows $i_1, i_2, \ldots, i_p$ of $Q$. For example, if

$$P = \begin{bmatrix} 1 & -2 & -2 & 4 \\ 2 & 3 & -1 & 2 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} -5 & 0 \\ 2 & 1 \\ -2 & 2 \\ 3 & 1 \end{bmatrix}.$$

Then for the major

$$\begin{vmatrix} -2 & 4 \\ 3 & 2 \end{vmatrix}$$

of $P$

$$\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$$

is the corresponding major of $Q$.

The **Binet-Cauchy theorem** is stated next. Proof of this theorem may be found in Hohn [3].

*THEOREM 7.16* *If $P$ is a $p \times q$ matrix and $Q$ is a $q \times p$ matrix, with $p \leq q$, then*

$$\det(PQ) = \sum(\text{product of the corresponding majors of } P \text{ and } Q).$$

*THEOREM 7.17* *Let $G$ be a connected undirected graph and $A$ an incidence matrix of a directed graph obtained by assigning orientations to the edges of $G$. Then*

$$\tau(G) = \det(AA^t) \tag{7.15}$$

*where $\tau(G)$ is the number of spanning trees of $G$.*

**PROOF 7.9** By the Binet-Cauchy theorem we have

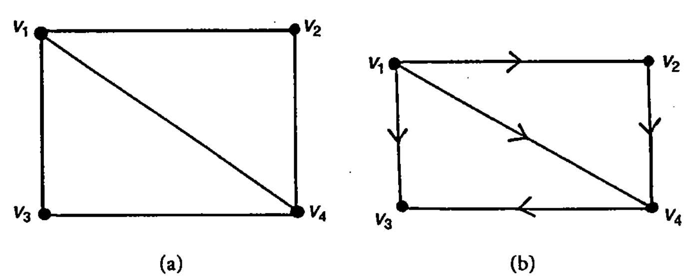
$$\det(AA^t) = \sum(\text{product of the corresponding majors of } A \text{ and } A^t) \tag{7.16}$$

Recall from Theorem 7.15 that a major of $A$ is nonzero iff the edges corresponding to the columns of the major form a spanning tree of $G$. Also, the corresponding majors of $A$ and $A^t$ have the same value equal to 0, 1, or $-1$ (Theorem 7.9). Thus, each nonzero term in the sum on the right-hand side of (7.16) has the value 1, and it corresponds to a spanning tree and vice versa. Hence the theorem. □

For example, consider the undirected graph $G$ shown in Fig. 7.11(a). Assigning arbitrary orientations to the edges of $G$, we obtain the directed graph in Fig. 7.11(b). If $A$ is the incidence matrix of this directed graph with vertex $v_4$ as reference vertex then it can be verified that

$$AA^t = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

and $\det(AA^t) = 8$. Thus, $G$ has eight spanning trees.




**7.11** (a) An undirected graph $G$; (b) directed graph obtained after assigning arbitrary orientations to the edges of $G$.

An interesting and useful interpretation of the matrix $AA^t$ now follows. Let $v_1, v_2, \ldots, v_n$ be the vertices of an undirected graph $G$. The **degree matrix** $K = [k_{ij}]$ of $G$ is an $n \times n$ matrix defined as follows.

$$k_{ij} = \begin{cases} -p, & \text{if } i \neq j \text{ and } p \text{ parallel edges connect } v_i \text{ and } v_j \\ d(v_i), & \text{if } i = j \end{cases}$$

We may easily verify that $K = A_c A_c^t$, and that it is independent of the choice of orientations for the edges of $G$. Also, if $v_i$ is the reference vertex, then $AA^t$ is obtained by removing row $i$ and column $i$ of $K$. In other words, $\det(AA^t)$ is the $(i, i)$ cofactor of $K$. It then follows from Theorem 7.17 that all the cofactors of $K$ are equal to the number of spanning trees of $G$. Thus, Theorem 7.17 may be stated in the following form originally presented by Kirchhoff [4].

***THEOREM 7.18*** *All the cofactors of the degree matrix of an undirected graph G have the same value equal to the number of spanning trees of G.*

Consider a connected undirected graph $G$. Let $A$ be the incidence matrix of $G$ with reference vertex $v_n$. Let $\tau_{i,n}$ denote the number of spanning 2-trees of $G$ such that the vertices $v_i$ and $v_n$ are in different components of these spanning 2-trees. Also, let $\tau_{ij,n}$ denote the number of spanning 2-trees such that vertices $v_i$ and $v_j$ are in the same component, and vertex $v_n$ is in a different component of these spanning 2-trees. If $\Delta_{ij}$ denotes the $(i, j)$ cofactor of $(AA^t)$, then we have the following result, proof of which may be found in [2].

***THEOREM 7.19*** *For a connected graph G,*

$$\tau_{i,n} = \Delta_{ii} \tag{7.17}$$

$$\tau_{ij,n} = \Delta_{ij} \tag{7.18}$$

Consider next a directed graph $G = (V, E)$ without self-loops and with $V = (v_1, v_2, \ldots, v_n)$. The **in-degree matrix** $K = [k_{ij}]$ of $G$ is an $(n \times n)$ matrix defined as follows:

$$k_{ij} = \begin{cases} -p, & \text{if } i \neq j \text{ and } p \text{ parallel edges are directed from } v_i \text{ to } v_j \\ d_{\text{in}}(v_i), & \text{if } i = j \end{cases}$$

The following result is due to Tutte [5]. Proof of this result may also be found in [2].

***THEOREM 7.20*** *Let K be the in-degree matrix of a directed graph G without self-loops. Let the ith row of K correspond to vertex $v_i$. Then the number $\tau_d$ of directed spanning trees of G having $v_r$ as root is given by*

$$\tau_d = \Delta_{rr} \tag{7.19}$$

*where $\Delta_{rr}$ is the $(r, r)$ co-factor of K.*

Note the similarity between Theorem 7.18 and Theorem 7.20.

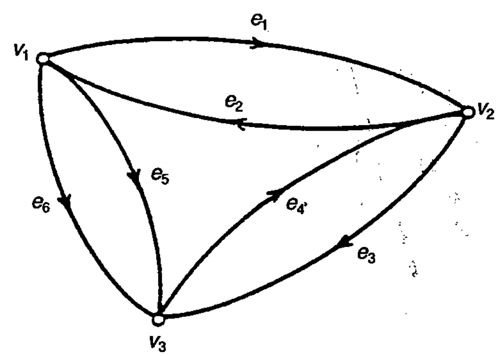
To illustrate Theorem 7.20, consider the directed graph $G$ shown in Fig. 7.12. The in-degree matrix $K$ of $G$ is

$$K = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

Then

$$\Delta_{11} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = 5$$

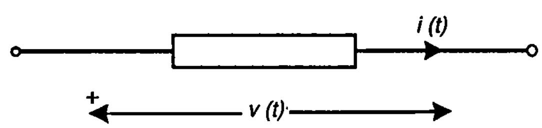
The five directed spanning trees of $G$ with vertex $v_1$ as root are $(e_1, e_5)$, $(e_1, e_6$, $(e_1, e_3)$, $(e_4, e_5)$, and $(e_4, e_6)$




7.12 A directed graph $G$.

## 7.7 Graphs and Electrical Networks

An electrical network is an interconnection of electrical network elements such as resistances, capacitances, inductances, voltage and current sources, etc. Each network element is associated with two variables, the voltage variable $v(t)$ and the current variable $i(t)$. We also assign reference directions to the network elements (see Fig. 7.13) so that $i(t)$ is positive whenever the current is in the direction of the arrow, and $v(t)$ is positive whenever the voltage drop in the network element is in the direction of the arrow. Replacing each element and its associated reference direction by a directed edge results in the directed graph representing the network. For example, a simple electrical network and the corresponding directed graph are shown in Fig. 7.14.




7.13 A network element with reference convention.

The physical relationship between the current and voltage variables of a network element is specified by Ohm's law. For voltage and current sources, the voltage and current variables are required to have specified values. The linear dependence among the voltage variables in the network and the linear dependence among the current variables are governed by Kirchhoff's voltage and current laws.

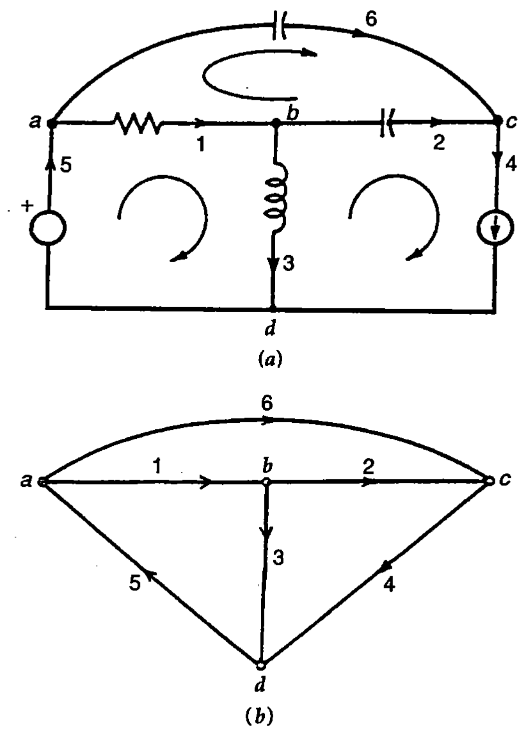
> **Kirchhoff's Voltage Law (KVL)**: The algebraic sum of the voltages around any circuit is equal to zero.
>
> **Kirchhoff's Current Law (KCL)**: The algebraic sum of the currents flowing out of a node is equal to zero.

As an example, the KVL equation for the circuit 1, 3, 5 and the KCL equation for the vertex $b$ in the graph of Fig. 7.14 are

$$\begin{array}{rl} \text{Circuit: } 1,3,5 & v_1 + v_3 + v_5 = 0 \\ \text{Vertex } b\text{:} & -i_1 + i_2 + i_3 = 0 \end{array}$$

It can easily be seen that KVL and KCL equations for an electrical network $N$ can be conveniently written as

$$A_c I_e = 0 \tag{7.20}$$




7.14 (a) An electrical network $N$; (b) directed graph representation of $N$.

and

$$B_c V_e = 0 \tag{7.21}$$

where $A_c$ and $B_c$ are, respectively, the incidence and circuit matrices of the directed graph representing $N$, and $I_e$ and $V_e$ are, respectively, the column vectors of element currents and voltages in $N$. Because each row in the cut matrix $Q_c$ can be expressed as a linear combination of the rows of the matrix, in (7.20) we can replace $A_c$ by $Q_c$. Thus, we have:

$$\text{KCL:} \quad Q_c I_e = 0 \tag{7.22}$$
$$\text{KVL:} \quad B_c V_e = 0 \tag{7.23}$$

From (7.22) we can see that KCL can also be stated as: The algebraic sum of the currents in any cut of $N$ is equal to zero.

If a network $N$ has $n$ vertices, $m$ elements, and its graph is connected then there are only $(n-1)$ linearly independent cuts and only $(m-n+1)$ linearly independent circuits (Theorem 7.13). Thus, in writing KVL and KCL equations we need to use only $B_f$, a fundamental circuit matrix and $Q_f$, a fundamental circuit matrix, respectively. Thus, we have

$$\text{KCL:} \quad Q_f I_e = 0 \tag{7.24}$$
$$\text{KVL:} \quad B_f V_e = 0 \tag{7.25}$$

We note that the KCL and the KVL equations depend only on the way the network elements are interconnected and not on the nature of the network elements. Thus, several results in electrical network theory are essentially graph theoretic in nature. Some of these results and their usefulness in electrical network analysis are presented in the remainder of this chapter. In the following a network $N$ and its directed graph are both denoted by $N$.

**THEOREM 7.21** *Consider an electrical network $N$. Let $T$ be a spanning tree of $N$, and let $B_f$ and $Q_f$ denote the fundamental circuit and the fundamental cutset matrices of $N$ with respect to $T$. If $I_e$ and $V_e$ are the column vectors of element currents and voltages and $I_c$ and $V_t$ are, respectively, the column vector of currents associated with the chords of $T$ and the column vector of voltages associated with the branches of $T$, then*

$$\text{Loop Transformation:} \quad I_e = B_f^t I_c \tag{7.26}$$

$$\text{Cutset Transformation:} \quad V_e = Q_f^t V_t \tag{7.27}$$

**PROOF 7.10** From Kirchhoff's laws we have

$$Q_f I_e = 0 \tag{7.28}$$

and

$$B_f V_e = 0 \tag{7.29}$$

Let us partition $I_e$ and $V_e$ as

$$I_e = \begin{bmatrix} I_c \\ I_t \end{bmatrix}$$

and

$$V_e = \begin{bmatrix} V_c \\ V_t \end{bmatrix}$$

where the vectors which correspond to the chords and branches of $T$ are distinguished by the subscripts $c$ and $t$, respectively. Then (7.28) and (7.29) can be written as

$$\begin{bmatrix} Q_{fc} & U \end{bmatrix} \begin{bmatrix} I_c \\ I_t \end{bmatrix} = 0 \tag{7.30}$$

and

$$\begin{bmatrix} U & B_{ft} \end{bmatrix} \begin{bmatrix} V_c \\ V_t \end{bmatrix} = 0 \tag{7.31}$$

Recall (7.11) that

$$B_{ft} = -Q_{fc}^t$$

Then we get from (7.30)

$$\begin{aligned} I_t &= -Q_{fc} I_c \\ &= B_{ft}^t I_c \end{aligned}$$

Thus

$$I_e = \begin{bmatrix} U \\ B_{ft}^t \end{bmatrix} I_c = B_f^t I_c$$

This establishes the loop transformation.

Starting from (7.31) we can show in a similar manner that

$$V_e = Q_f^t V_t$$

thereby establishing the cutset transformation. □

In the special case in which the incidence matrix $A$ is used in place of the fundamental cutset matrix, the cutset transformation (7.27) is called the **node transformation**. The loop, cutset, and node transformations have been extensively employed to develop different methods of network analysis. The loop method of analysis develops a system of network equations which involve only the chord currents as variables. The cutset (node) method of analysis develops a system of equations involving only the branch (node) voltages as variables. Thus, the loop and cutset (node) methods result in systems of equations involving $m - n + 1$ and $n - 1$ variables, respectively. In the mixed-variable method of analysis, which is essentially a combination of both the loop and cutset methods, some of the independent variables are currents and the others are voltages. The minimum number of variables required in the mixed-variable method of analysis is determined by what is known as the principal partition of a graph introduced by Kishi and Kajitani in a classic paper [6]. Ohtsuki, Ishizaki, and Watanabe [7] discuss several issues relating to the mixed-variable method of analysis. A detailed discussion of the principal partition of a graph and the different methods of network analysis including the state-variable method may be found in [1].

## 7.8 Tellegen's Theorem and Network Sensitivity Computation

Here we first present a simple and elegant theorem due to Tellegen [8]. The proof of this theorem is essentially graph theoretic in nature and is based on the loop and the cutset transformations, (7.26) and (7.27), and the orthogonality relation (Theorem 7.12). Using this theorem we develop the concept of the adjoint of a network and its application in network sensitivity computations.

*THEOREM 7.22* *Consider two electrical networks $N$ and $\hat{N}$ such that the graphs associated with them are identical. Let $V_e$ and $\psi_e$ denote the element voltage vectors of $N$ and $\hat{N}$, respectively, and let $I_e$ and $\Lambda_e$ be the corresponding element current vectors. Then*

$$\begin{aligned} V_e^t \Lambda_e &= 0 \\ I_e^t \psi_e &= 0 \end{aligned}$$

**PROOF 7.11** If $B_f$ and $Q_f$ are the fundamental circuit and cutset matrices of $N$ (and hence also of $\hat{N}$), then from the loop and cutset transformations we obtain

$$V_e = Q_f^t V_t$$

and

$$\Lambda_e = B_f^t \Lambda_c$$

So

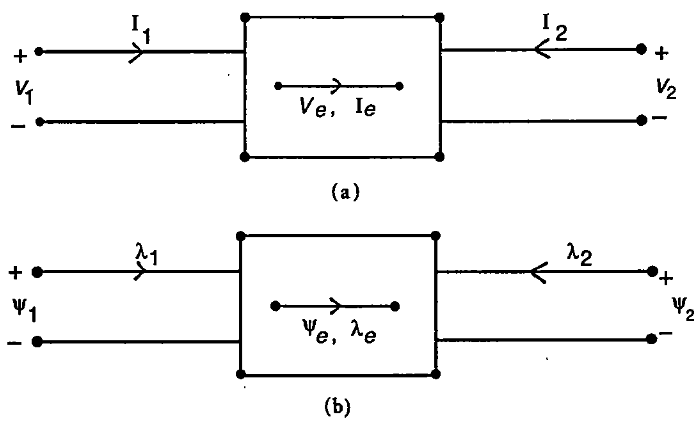
$$\begin{aligned} V_e^t \Lambda_e &= V_t^t (Q_f B_f^t) \Lambda_c \\ &= 0, \quad \text{by Theorem 12} \end{aligned}$$

Proof follows in a similar manner. □

The adjoint network was introduced by Director and Rohrer [9], and our discussion is based on their work. A more detailed discussion may be found in [1].

Consider a lumped, linear time-invariant network $N$. We assume, without loss of generality, that $N$ is a 2-port network. Let $\hat{N}$ be a 2-port network which is topologically equivalent to $N$. In other words, the graph of $\hat{N}$ is identical to that of $N$. The corresponding elements of $N$ and $\hat{N}$ are denoted by the same symbol. Our goal now is to define the elements of $\hat{N}$ so that $\hat{N}$ in conjunction with $N$ can be used in computing the sensitivities of network functions of $N$.

Let $V_e$ and $I_e$ denote, respectively, the voltage and the current associated with the element $e$ in $N$, and $\psi_e$ and $\lambda_e$ denote, respectively, the voltage and the current associated with the corresponding element $e$ in $\hat{N}$. Also, $V_i$ and $I_i$, $i = 1, 2$, denote the voltage and current variables associated with the ports of $N$, and $\psi_i$ and $\lambda_i$, $i = 1, 2$, denote the corresponding variables for the ports of $\hat{N}$ (see Fig. 7.15).




7.15 (a) A 2-port network $N$; (b) adjoint network $\hat{N}$ of N.

Applying Tellegen's theorem to $N$ and $\hat{N}$ we get

$$V_1\lambda_1 + V_2\lambda_2 = \sum_e V_e\lambda_e \tag{7.32}$$

and

$$I_1\psi_1 + I_2\psi_2 = \sum_e I_e\psi_e \tag{7.33}$$

Suppose we now perturb the values of elements of $N$ and apply Tellegen's theorem to $\hat{N}$ and the perturbed network $N$:

$$(V_1 + \Delta V_1)\lambda_1 + (V_2 + \Delta V_2)\lambda_2 = \sum_e (V_e + \Delta V_e)\lambda_e \tag{7.34}$$

and

$$(I_1 + \Delta I_1)\psi_1 + (I_2 + \Delta I_2)\psi_2 = \sum_e (I_e + \Delta I_e)\psi_e \tag{7.35}$$

where $\Delta V$ and $\Delta I$ represent the changes in the voltage and current which result as a consequence of the perturbation of the element values in $N$. Subtracting (7.32) from (7.34) and subtracting (7.33) from (7.35)

$$\Delta V_1\lambda_1 + \Delta V_2\lambda_2 = \sum_e \Delta V_e\lambda_e \tag{7.36}$$

and

$$\Delta I_1\psi_1 + \Delta I_2\psi_2 = \sum_e \Delta I_e\psi_e \tag{7.37}$$

Subtracting (7.37) from (7.36) yields

$$(\Delta V_1\lambda_1 - \Delta I_1\psi_1) + (\Delta V_2\lambda_2 - \Delta I_2\psi_2) = \sum_e (\Delta V_e\lambda_e - \Delta I_e\psi_e) \tag{7.38}$$

We wish to define the corresponding element of $\hat{N}$ for every element in $N$ so that each term in the summation on the right-hand side of (7.38) reduces to a function of the voltage and current variables and the change in value of the corresponding network element. We illustrate this for resistance elements. Consider a resistance element $R$ in $N$. For this element we have

$$V_R = RI_R \tag{7.39}$$

Suppose we change $R$ to $\Delta R$, then

$$(V_R + \Delta V_R) = (R + \Delta R)(I_R + \Delta I_R) \tag{7.40}$$

Neglecting second-order terms, (7.40) simplifies to

$$V_R + \Delta V_R = RI_R + R\Delta I_R + I_R\Delta R \tag{7.41}$$

Subtracting (7.39) from (7.41),

$$\Delta V_R = R\Delta I_R + I_R\Delta R \tag{7.42}$$

Now using (7.42) the terms in (7.38) corresponding to the resistance elements of $N$ can be written as

$$\sum_R [R\lambda_R - \psi_R]\,\Delta I_R + I_R\lambda_R\Delta R \tag{7.43}$$

If we now choose

$$\psi_R = R\lambda_R \tag{7.44}$$

then (7.43) reduces to

$$\sum_R I_R\lambda_R\Delta R \tag{7.45}$$

which involves only the network variables in $N$ (before perturbation) and $\hat{N}$ and the changes in resistance values. Equation (7.44) is the relation for a resistance. Therefore, the element in $\hat{N}$ corresponding to a resistance element of value $R$ in $N$ is also a resistance of value $R$.

Proceeding in a similar manner we can determine the elements of $\hat{N}$ corresponding to other types of network elements (inductance, capacitance, controlled sources, etc.) The network $\hat{N}$ so obtained is called the **adjoint** of $N$. A table defining adjoint elements corresponding to different types of network elements may be found in [1].

We now illustrate the application of the adjoint network in the computation of the sensitivity of a network function. Note that the sensitivity of a network function $F$ with respect to a parameter $x$ is a measure of the effect on $F$ of an incremental change in $x$. Computing this sensitivity essentially involves determining $\partial F/\partial x$.

For the sake of simplicity consider the resistance network shown in Fig. 7.16(a). Let us assume that resistance $R$ is perturbed from its nominal value of 3 Ω. Assume that no changes occur in the values of the other resistance elements. We wish to compute $\partial F/\partial R$ where $F$ is the open-circuit voltage ratio, that is,

$$F = \left.\frac{V_2}{V_1}\right|_{I_2=0}$$

In other words, to compute $F$, we connect a voltage source of value $V_1 = 1$ across port 1 of $N$ and open-circuit port 2 of $N$ (so that $I_2 = 0$). So, $\Delta V_1 = 0$ and $\Delta I_2 = 0$ and (7.38) reduces to

$$-\Delta I_1\psi_1 + \Delta V_2\lambda_2 = I_R\lambda_R\Delta R \tag{7.46}$$

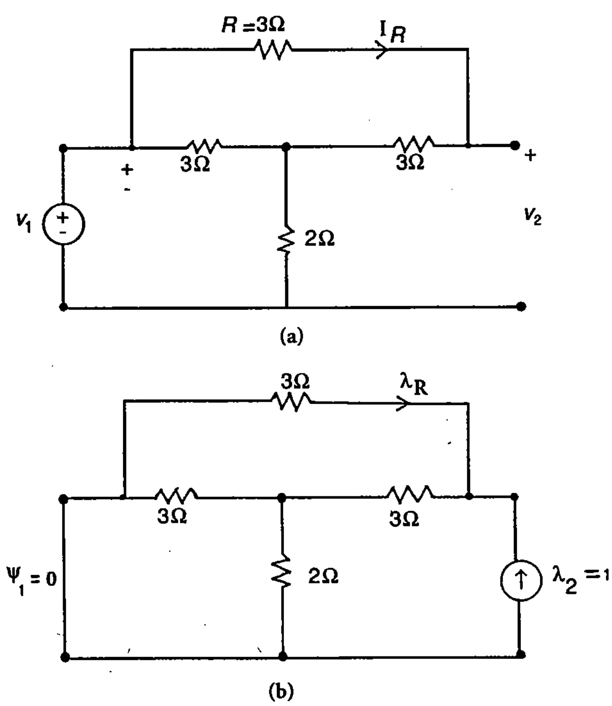
Now we need to determine $\Delta V_2$ as a function of $\Delta R$. This could be achieved if we set $\psi_1 = 0$ and $\lambda_2 = 1$ for the adjoint network $\hat{N}$. Connect a current source of value $\lambda_2 = 1$ across port 2 and short circuit port 1 of $\hat{N}$. The resulting adjoint network is shown in Fig. 7.16(b). With port variables of $\hat{N}$ defined as above, (7.46) reduces to

$$\Delta V_2 = I_R\lambda_R\Delta_R$$

Thus,

$$\partial F/\partial R = \partial V_2/\partial R = I_R\lambda_R$$

where $I_R$ and $\lambda_R$ are the currents in the networks $N$ and $\hat{N}$ shown in Fig. 7.16. Thus, in general, computing the sensitivity of a network function essentially reduces to the analysis of $N$ and $\hat{N}$ under appropriate excitations at their ports. Note that we do not need to express the network function explicitly in terms of the network elements nor do we need to calculate partial derivatives.




**7.16** (a) A 2-port network $N$; (b) adjoint network $\hat{N}$.

For the example under consideration, we calculate $I_R = 1/12$ A and $\lambda_R = -7/12$ A with the result that $\partial F/\partial R = -7/144$. A further discussion of the adjoint network and related results may be found in Section 7.3.

## 7.9 Arc Coloring Theorem and the No-Gain Property

We now derive a profound result in graph theory, the arc coloring theorem for directed graphs, and discuss its application in establishing the no-gain property of resistance networks. In the special case of undirected graphs the arc coloring theorem reduces to the "painting" theorem. Both of these theorems (Minty [10]) are based on the notion of **painting a graph**.

Given an undirected graph with edge set $E$, a painting of the graph is a partitioning of $E$ into three subsets, $R$, $G$, and $B$, such that $|G| = 1$. We may consider the edges in the set $R$ as being "painted red," the edge in $G$ as being "painted green" and the edges in $B$ as being "painted blue."

*THEOREM 7.23* *For any painting of a graph, there exists a circuit $C$ consisting of the green edge and no blue edges, or a cutset $C^*$ consisting of the green edge and no red edges.*

**PROOF 7.12** Consider a painting of the edge set $E$ of a graph $G$. Assuming that there does not exist a required circuit, we shall establish the existence of a required cutset.

Let $E' = R \cup G$ and $T'$ denote a spanning forest of the subgraph induced by $E'$, containing the green edge. (Note that the subgraph induced by $E'$ may not be connected). Then construct a spanning tree $T$ of $G$ such that $T' \subseteq T$.

Now consider any red edge $y$ which is not in $T'$, and hence not in $T$. Because the fundamental circuit of $y$ with respect to $T$ is the same as the fundamental circuit of $y$ with respect to $T'$, this circuit consists of no blue edges. Furthermore, this circuit will not contain the green edge, for otherwise a circuit consisting of the green edge and no blue edges would exist contrary to our assumption. Thus, the fundamental circuit of a red edge with respect to $T$ does not contain the green edge. Then it follows from Theorem 8 that the fundamental cutset of the green edge with respect to $T$ contains no red edges. Thus, this cutset satisfies the requirements of the theorem. □

A painting of a directed graph with edge set $E$ is a partitioning of $E$ into three sets $R$, $G$, and $B$, and the distinguishing of one element of the set $G$. Again, we may regard the edges of the graph as being colored red, green, or blue with exactly one edge of $G$ being colored dark green. Note that the dark green edge is also to be treated as a green edge.

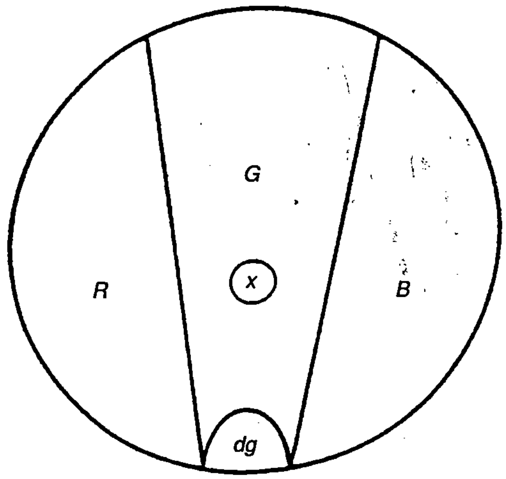
Next we state and prove Minty's arc coloring theorem.

*THEOREM 7.24* *For any painting of a directed graph exactly one of the following is true:*

1. *A circuit exists containing the dark green edge, but no blue edges, in which all the green edges are similarly oriented.*
2. *A cutset exists containing the dark green edge, but no red edges, in which all the green edges are similarly oriented.*

**PROOF 7.13** Proof is by induction on the number of green edges. If only one green edge exists, then the result will follow from Theorem 7.23. Assume then that the result is true when the number of green edges is $m \geq 1$. Consider a painting in which $m + 1$ edges are colored green. Pick a green edge $x$ other than the dark green edge (see Fig. 7.17). Color the edge $x$ red. In the resulting painting we find $m$ green edges. If a cutset of type 2 is now found, then the theorem is proved. On the other hand if we color the edge $x$ blue and in the resulting painting a circuit of type 1 exists, then the theorem is proved.

Suppose neither occurs. Then, using the induction hypothesis we have the following:




**7.17** Painting of a directed graph.

1. A cutset of type 2 exists when $x$ is colored blue.
2. A circuit of type 1 exists when $x$ is colored red.

Now let the corresponding rows of the circuit and cutset matrices be

| | $dg$ | $R$ | $B$ | $G$ | $x$ |
|---|---|---|---|---|---|
| Cutset | $+1$ | $00\ldots0\ 0$ | $1-1\ldots01$ | $111\ldots0$ | ? |
| Circuit | $+1$ | $-11\ldots0-1$ | $0\ 0\ldots00$ | $011\ldots0$ | ? |

Here we have assumed, without loss of generality, that +1 appears in the dark green position of both rows.

By the orthogonality relation (Theorem 7.12) the inner product of these two row vectors is zero. No contribution is made to this inner product from the red edges or from the blue edges. The contribution from the green edges is a non-negative integer $p$. The dark green edge contributes 1 and the edge $x$ contributes an unknown integer $q$ which is 0, 1, or $-1$. Thus, we have $1+p+q=0$. This equation is satisfied only for $p=0$ and $q=-1$. Therefore, in one of the rows, the question mark is $+1$ and in the other it is $-1$. The row in which the question mark is 1 corresponds to the required circuit or cutset. Thus, either statement 1 or 2 of the theorem occurs. Both cannot occur simultaneously because the inner product of the corresponding circuit and cutset vectors will then be nonzero. □

*THEOREM 7.25 Each edge of a directed graph belongs to either a directed circuit or to a directed cutset but no edge belongs to both. (Note: A cutset is a* **directed cutset** *if all its edges are similarly oriented.)*

**PROOF 7.14** Proof will follow if we apply the arc coloring theorem to a painting in which all the edges are colored green and the given edge is colored dark green. □

We next present an application of the arc coloring theorem in the study of electrical networks. We prove what is known as the **no-gain property** of resistance networks. Our proof is the result of the work of Wolaver [11] and is purely graph theoretic in nature.

*THEOREM 7.26 In a network of sources and (linear/nonlinear) positive resistances the magni-*

*tude of the current through any resistance with nonzero voltage is not greater than the sum of the magnitudes of the currents through the sources.*

**PROOF 7.15** Let us eliminate all the elements with zero voltage by considering them to be short-circuits and then assign element reference directions so that all element voltages are positive.

Consider a resistance with nonzero voltage. Thus, no directed circuit can contain this resistance, for if such a directed circuit were present, the sum of all the voltages in the circuit would be nonzero, contrary to Kirchhoff's voltage law. It then follows from Theorem 7.25 that a directed cutset contains the resistance under consideration.

Pick a directed cutset that contains the considered resistance. Let the current through this resistance be $i_o$. Let $R$ be the set of all other resistances in this cutset and let $S$ be the set of all sources. Then, applying Kirchhoff's current law to the cutset, we obtain

$$i_o + \sum_{k\in R} i_k + \sum_{s\in S} \pm i_s = 0 \tag{7.47}$$

Because all the resistances and voltages are positive, every resistance current is positive. Therefore, we can write the above equation as

$$|i_o| + \sum_{k\in R} |i_k| + \sum_{s\in S} \pm i_s = 0 \tag{7.48}$$

and so

$$|i_o| \le \sum_{s\in S} \mp i_s \le \sum_{s\in S} |i_s| \tag{7.49}$$

Thus follows the theorem. □

The following result is the dual of the above theorem. Proof of this theorem follows in an exactly dual manner, if we replace current with voltage, voltage with current, and circuit with cutset in the proof of the above theorem.

***THEOREM 7.27*** *In a network of sources and (linear/nonlinear) positive resistances, the magnitude of the voltage across any resistance is not greater than the sum of the voltages across all the sources.*

Chua and Green [12] used the arc-coloring theorem to establish several properties of nonlinear networks and nonlinear multiport resistive networks.

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