6.4 GRAPHS AND VECTOR SPACES

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INTRODUCTION

Electrical circuit theory is one of the earliest applications of graph theory to a problem in physical science. The dynamic behavior of an electrical circuit is governed by three laws: Kirchhoff's voltage law, Kirchhoff's current law and Ohm's law. Each element in a circuit is associated with two variables, namely, the current variable and the voltage variable. Kirchhoff's voltage law requires that the algebraic sum of the voltages around a circuit is zero, and Kirchhoff's current law requires that the algebraic sum of the currents across a cut is zero. Thus, circuits and cuts define a linear relationship among the voltage variables and a linear relationship among the current variables, respectively. It is for this reason that circuits, cuts, and the vector spaces associated with them have played a major role in the discovery of several fundamental properties of electrical circuits arising from the structure or the interconnection of the circuit elements. Several graph theorists and circuit theorists have immensely contributed to the development of what we may now call the structural theory of electrical circuits. The significance of the results to be presented in this section goes well beyond their application to circuit theory. They will bring out the fundamental duality that exists between circuits and cuts and the influence of this duality on the structural theory of graphs. Most of the results in this section are also relevant to the development of combinatorial optimization theory as well as matroid theory.

6.4.1 Basic Concepts and Definitions

Although the terms *node* and *oriented graph* are commonly used in electrical circuit theory, we use the terms *vertex* and *directed graph* along with all the other basic terminology of graph theory established in Chapter 1.

For the sake of completeness, we begin with a review of certain basic concepts and definitions. For concepts not discussed here, the reader is referred to [GrYe99] and [ThSw92].

NOTATION: Unless otherwise specified, $G = (V, E)$ is a graph (or digraph) with n vertices, $V = \{v_1, v_2, \ldots, v_n\}$, and m edges, $E = \{e_1, e_2, \ldots e_m\}$.

If vertices v_i and v_j are the endpoints (or end vertices) of an edge then, when there is no ambiguity, we denote that edge by the ordered pair (v_i, v_j).

DEFINITIONS

D1: A graph is called a ***trivial graph*** if it has only one vertex and no edge. A graph with no edges is called an ***empty graph***. A graph with no vertices and hence no edges is called a ***null graph*** and will be denoted by $\emptyset$.

REMARK

R1: In this section we consider only graphs in which all edges have two distinct endpoints (i.e., no self-loops).

EXAMPLE

E1: Examples 1 through 9 in this section refer to the graph shown in Figure 6.4.1.

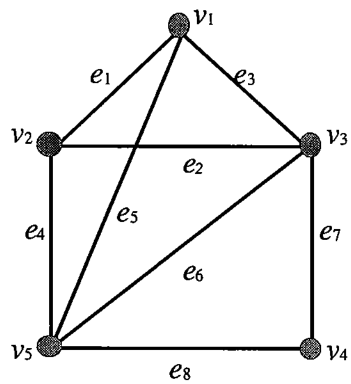


Figure 6.4.1

Subgraphs and Complements

DEFINITIONS

D2: A graph $G' = (V', E')$ is called a ***subgraph*** of graph $G = (V, E)$ if $V' \subseteq V$, $E' \subseteq E$ and V' contains all the endpoints of all the edges in E'.

D3: Each subset $E' \subseteq E$ defines a unique subgraph $G' = (V', E')$ of graph $G = (V, E)$, where V' consists of only those vertices which are the endpoints of the edges in E'. The subgraph G' is called the ***induced subgraph of G on the edge set E'***. Note that an edge-induced subgraph will not have isolated vertices.

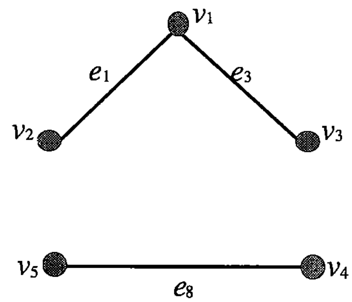
D4: Each subset $V' \subseteq V$ defines a unique subgraph $G' = (V', E')$ of graph $G = (V, E)$, where E' consists of those edges whose endpoints are in V'. The subgraph G' is called the ***induced subgraph of G on the vertex set V'***. Note that a vertex-induced subgraph may have isolated vertices.

D5: Given a subgraph $G' = (V', E')$ of graph $G = (V, E)$, the subgraph $G'' = (V, E -$

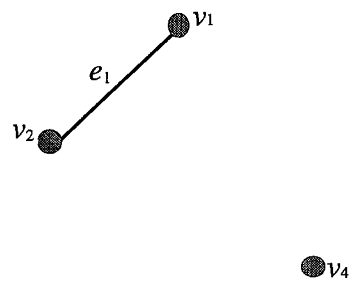
E') is called the (***edge-*)*complement of G′ in G***.

EXAMPLES

E2: For the set $E' = \{e_1, e_3, e_8\}$, the corresponding edge-induced subgraph of graph G in Figure 6.4.1 is shown in Figure 6.4.2 (a). For the set $V' = \{v_1, v_2, v_4\}$, the corresponding vertex-induced subgraph of G is shown in Figure 6.4.2 (b).



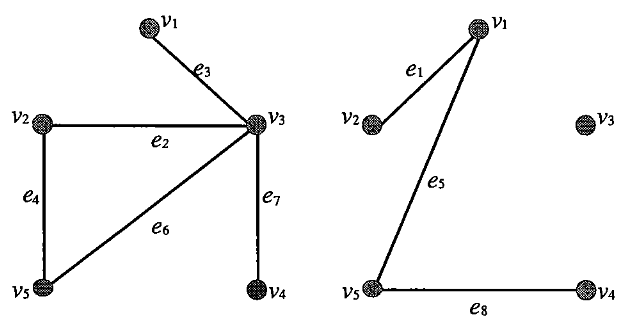
(a) An edge- induced subgraph of the graph G



(b) A vertex-induced subgraph of the graph G

Figure 6.4.2 **An edge-induced subgraph and a vertex-induced subgraph.**

E3: The complement of the subgraph G' of Figure 6.4.3(a) in the graph G of Figure 6.4.1 is shown in Figure 6.4.3(b).



(a) Subgraph G' (b) Complement of G' in the graph G

Figure 6.4.3 **A subgraph G' and its complement in G.**

Components, Spanning Trees, and Cospanning Trees

DEFINITIONS

D6: A ***closed trail*** is a closed walk with no repeated edges.

TERMINOLOGY: A closed trail is also called a *circ*, which we formally state in Definition 20 and use thereafter.

D7: A ***circuit*** is a closed trail with no repeated vertices except the initial and terminal ones.

TERMINOLOGY: Several authors use the term *cycle* instead of circuit. In electrical circuit literature, the term circuit is commonly understood as defined in Definition 7.

D8: A graph G is ***connected*** if there is a path between every pair of vertices of G.

D9: A maximal connected subgraph of a graph is called a ***component*** of the graph. An isolated vertex is by itself considered a single component.

D10: A ***tree*** of a graph is a connected subgraph of the graph containing no circuits. If a tree of a connected graph G contains all the vertices of G then it is called a ***spanning tree*** of G. The complement of a spanning tree T in G is called a ***cospanning tree*** of G.

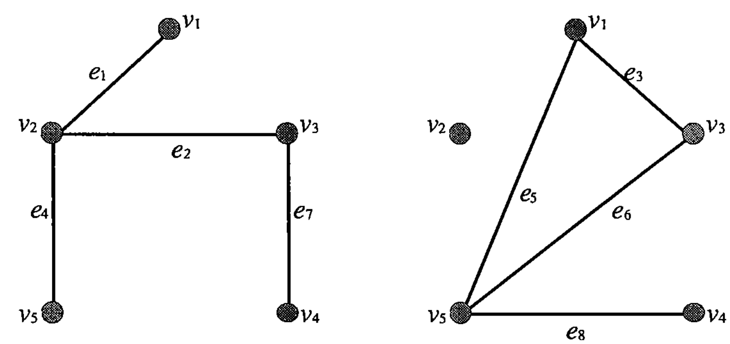
D11: A ***spanning forest*** of a non-connected graph G with p components is a collection of p spanning trees, one for each component.

D12: The edges of a spanning tree T are called the ***branches*** of T. The edges of a cospanning tree are called the ***chords*** of the spanning tree.

D13: Let G be an n-vertex graph with m edges and p components. The ***rank*** $\rho(G)$ and ***nullity*** $\mu(G)$ of G are given by $\rho(G) = n - p$ and $\mu(G) = m - n + p$.

EXAMPLE

E4: A spanning tree T and the corresponding cospanning tree of the connected graph G of Figure 6.4.1 are shown in Figure 6.4.4.



(a) A spanning tree T of G (b) The cospanning tree with repect to T

Figure 6.4.4 A spanning tree and corresponding cospanning tree of graph G.

FACTS

F1: There is exactly one path between any two vertices of a spanning tree.

F2: A spanning tree of a connected n-vertex graph has $n-1$ branches and a cospanning tree has $m-n+1$ chords. A spanning forest of a graph having p components has $n-p$ branches and $m-n+p$ chords.

REMARK

R2: Unless stated otherwise, all graphs G considered in this section are connected.

Cuts and Cutsets

DEFINITIONS

D14: Consider a connected graph $G=(V,E)$. Let V_1 and V_2 be two disjoint subsets of V such that $V=V_1\cup V_2$ (i.e., V_1 and V_2 form a *partition* of V). Then the set of all those edges of G having one end vertex in V_1 and the other in V_2 is called a ***cut*** of G. This cut is denoted as $\langle V_1,V_2\rangle$. The set of edges incident on a vertex forms a cut, and is called an ***incidence set***.

D15: Removal of the edges in a cut from a connected graph G will disconnect the graph. In other words, the resulting graph will have at least two components. A cut of a connected graph is called a ***cutset*** if the removal of the edges in the cut results in a non-connected graph with *exactly* two components. Equivalently, a cutset of a connected graph is a minimal set of edges whose removal disconnects the graph.

EXAMPLE

E5: For the graph G in Figure 6.4.1, the cut $\langle V_1,V_2\rangle$, where $V_1=\{v_1,v_3,v_5\}$ and $V_2=\{v_2,v_4\}$ consists of the edges e_1,e_2,e_4,e_7, and e_8, and it is shown in Figure 6.4.5(a). Removing these edges results in a non-connected graph with three components. So, $\langle V_1,V_2\rangle$ is not a cutset. A cutset consisting of the edges e_4,e_5,e_6, and e_7 is shown in Figure 6.4.5(b). Removing these edges results in a non-connected graph with two components.

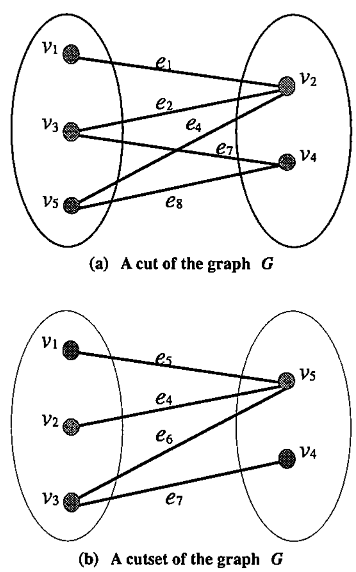


Figure 6.4.5 A cut and a cutset of the graph G.

The Vector Space of a Graph Under Ring Sum of Its Edge Subsets

DEFINITIONS

D16: Suppose the edge-set of a graph G is $E = \{e_1, e_2, e_3, \ldots, e_m\}$. Then each subset E' of E can be represented by a ***binary m-vector*** in which the ith component is 1 if and only if the edge e_i is in E'. For example, the binary vector $(1,0,0,1,1,1,0,0)$ represents the edge subset $\{e_1, e_4, e_5, e_6\}$ of the graph G of Figure 6.4.1.

D17: The ***ring sum*** (or ***symmetric difference***) of two sets E_1 andE_2, denoted as $E_1 \oplus E_2$, is the set of those edges which belong to E_1 or to E_2 but not to both E_1 and E_2.

D18: The ***ring sum*** of two m-vectors $X = (x_1, x_2, x_3, \ldots, x_i, \ldots, x_m)$ and $Y = (y_1, y_2, y_3, \ldots, y_i, \ldots, y_m)$ is the vector $Z = (z_1, z_2, z_3, \ldots, z_i, \ldots, z_m)$, where $z_i = x_i \otimes y_i$ and $\otimes$ is the logical *exclusive-or* operation (i.e., $1 \otimes 0 = 1$; $0 \otimes 1 = 1$; $0 \otimes 0 = 0$; and $1 \otimes 1 = 0$).

FACTS

F3: The m-vector representing the ring sum of two subsets of edges is the ring sum of the m-vectors representing these edge subsets. The set of m-vectors representing all the 2^m edge subsets of a graph G (including the null set) forms an m-dimensional ***vector space*** over GF(2), the field of integers modulo 2, under the ring sum operation $\oplus$.

NOTATION: This vector space of edge subsets of a graph G (and hence of the corresponding edge-induced subgraphs of G) is denoted by $\Psi(G)$.

REMARKS

R3: Throughout this section all vectors are assumed to be row vectors.

R4: In this section an edge subset is used to refer to the corresponding edge-induced subgraph. The vector space $\Psi(G)$ will be used to denote the vector space of all binary m-vectors as well as the vector space of all edge-induced subgraphs of G. Observe that the null set (or null graph $\emptyset$) is the 0-vector of $\Psi(G)$.

R5: In electrical engineering literature, a cut is also referred to as a ***seg*** [Re61].

R6: Proofs of most results in this section may be found in standard texts [SeRe61], [Ch71b], [De74], [ThSw92], and [SwTh81].

6.4.2 The Circuit Subspace in an Undirected Graph

DEFINITIONS

D19: A graph is ***even*** if the degree of every vertex in the graph is even. Clearly, a circuit is an even graph.

D20: A ***circ*** of a graph is a closed trail. The null graph is considered as a circ.

NOTATION: The set of all circs of a graph G is denoted by $\hat{C}(G)$. In other words, $\hat{C}(G)$ is the set of all circuits and unions of edge-disjoint circuits of the graph G (including the null graph $\emptyset$).

FACTS

F4: A subgraph of a graph is a circ if and only if it is even.

F5: A circ is a circuit or union of edge-disjoint circuits. Thus, the edge set of an even graph can be partitioned into edge subsets such that each subset in the partition forms a circuit.

F6: The ring sum of any two even subgraphs of a graph is even. Thus, the set $\hat{C}(G)$ is closed under ring sum.

F7: $\hat{C}(G)$ is a subspace of the vector space $\Psi(G)$ and is called the ***circuit subspace*** of G.

EXAMPLE

E6: Two circs of the graph G of Figure 6.4.1 and their ring sum, which is clearly a circ, are shown in Figure 6.4.6.

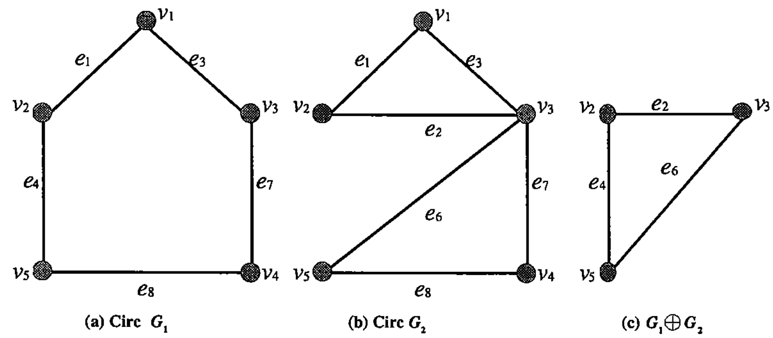


Figure 6.4.6 **Two circs of the graph G and their ring sum.**

REMARKS

R7: Fact 5 is attributed to Veblen [Ve31].

R8: A connected, even graph G is *eulerian*, i.e., there exists a circ that contains all the edges of G (see Section 4.2).

Fundamental Circuits and the Dimension of the Circuit Subspace

DEFINITIONS

D21: Let T be a spanning tree of a connected graph G. Adding a chord c to T produces a unique circuit called the ***fundamental circuit*** of G with respect to chord c.

NOTATION: If e_i is a chord of a spanning tree T, then C_i will denote the fundamental circuit with respect to e_i.

FACTS

F8: Given a connected graph G and a spanning tree T, there are $m-n+1$ fundamental circuits, one for each chord of T.

F9: The fundamental circuit with respect to chord c contains only one chord of the spanning tree T, namely, the chord c. The chord c is not present in any other fundamental circuit with respect to T.

F10: The $(m - n + 1)$ fundamental circuits with respect to a spanning tree of a connected graph G are linearly independent in the circuit subspace $\hat{C}(G)$.

F11: If a circ of a graph G contains the chords $e_a, e_b, \ldots, e_k$, then the circ can be expressed as the ring sum of the fundamental circuits $C_a, C_b, \ldots, C_k$.

F12: The fundamental circuits with respect to a spanning tree of a connected graph G constitute a basis for the circuit subspace $\hat{C}(G)$, and hence, the dimension of $\hat{C}(G)$ is equal to $m - n + 1$, the nullity $\mu(G)$.

F13: The dimension of the circuit subspace $\hat{C}(G)$ of a graph having p components is equal to $\mu(G) = m - n + p$.

EXAMPLE

E7: The set of fundamental circuits with respect to the spanning tree $T = \{e_1, e_2, e_4, e_7\}$ of the graph shown in Figure 6.4.1 is

Chord e_3	$C_3 = \{e_3, e_1, e_2\}$
Chord e_5	$C_5 = \{e_5, e_1, e_4\}$
Chord e_6	$C_6 = \{e_6, e_2, e_4\}$
Chord e_8	$C_8 = \{e_8, e_2, e_4, e_7\}$

It can be verified that the circ $\{e_1, e_4, e_5, e_6, e_7, e_8\}$, which contains the chords e_5, e_6 and e_8, is the ring sum of the fundamental circuits C_5, C_6, and C_8. This illustrates Fact 11.

6.4.3 The Cutset Subspace in an Undirected Graph

Recall from the definitions in Section 6.4.1 that a cutset is also a cut. Several facts that highlight the duality between cuts and circs will be presented next.

NOTATION: The collection of all cutsets and unions of edge-disjoint cutsets of a graph G is denoted by $\lambda(G)$. The null graph $\emptyset$ is considered a cut and hence belongs to $\lambda(G)$.

FACTS

F14: A cut of a connected graph G is the union of some edge-disjoint cutsets of G. Thus, $\lambda(G)$ is the collection of cuts of G.

F15: The ring sum of any two cuts of a graph G is also a cut of G, i.e., $\lambda(G)$ is closed under ring sum.

F16: $\lambda(G)$ is a subspace of the vector space $\Psi(G)$ and is called the ***cutset subspace*** of G.

EXAMPLE

E8: Consider the graph in Figure 6.4.1 and the cuts $S_1 = \langle V_1, V_2 \rangle$ and $S_2 = \langle V_3, V_4 \rangle$ in Figure 6.4.5, where $V_1 = \{v_1, v_3, v_5\}$, $V_2 = \{v_2, v_4\}$, $V_3 = \{v_1, v_2, v_3\}$ and $V_4 = \{v_4, v_5\}$. Then $S_1 = \{e_1, e_2, e_4, e_7, e_8\}$, $S_2 = \{e_4, e_5, e_6, e_7\}$, and $S_1 \oplus S_2 = \{e_1, e_2, e_5, e_6, e_8\}$.

Moreover, it can be seen that $S_1 \oplus S_2 = \langle A \cup D, B \cup C \rangle$, where

$A = V_1 \cap V_3 = \{v_1, v_3\}$,

$B = V_1 \cap V_4 = \{v_5\}$,

$C = V_2 \cap V_3 = \{v_2\}$,

$D = V_2 \cap V_4 = \{v_4\}$.

In fact, this illustration is also the basis of the proof of Fact 15.

Fundamental Cutsets and the Dimension of the Cutset Subspace

DEFINITIONS

D22: Let T be a spanning tree of a connected graph G, and let b be a branch of T. If V_1 and V_2 are the vertex-sets of the two components of $T - b$, then we can verify that the cut $\langle V_1, V_2 \rangle$ is a cutset of G. This cutset is called the ***fundamental cutset*** of G with respect to the branch b of T.

NOTATION: If e_i is a branch of a spanning tree T, then S_i denotes the fundamental cutset with respect to the branch e_i.

D23: An ***incidence set*** of a vertex v in a graph G is the cut consisting of the set of edges of G that are incident on v.

FACTS

F17: Given a connected graph G and a spanning tree T, there are $n - 1$ fundamental cutsets, one for each branch of T.

F18: The fundamental cutset with respect to branch b of a spanning tree T contains only one branch, namely, the branch b. The branch b is not present in any other fundamental cutset with respect to T.

F19: The $n - 1$ fundamental cutsets with respect to a spanning tree of a connected n-vertex graph G are linearly independent in the cutset subspace $\lambda(G)$.

F20: If a cut of a graph G contains the branches $e_a, e_b, \ldots e_k$, then the cut can be expressed as the ring sum of the fundamental cutsets $S_a, S_b, \ldots, S_k$.

F21: The fundamental cutsets with respect to a spanning tree of a connected graph G constitute a basis for the cutset subspace $\lambda(G)$ of G, and hence the dimension of $\lambda(G)$ is equal to $n - 1$, the rank $\rho(G)$.

F22: The dimension of the cutset subspace $\lambda(G)$ of a graph having p components is equal to $\rho(G) = n - p$.

F23: The incidence sets of any $n - 1$ vertices of a connected n-vertex graph G form a basis of the cutset subspace $\lambda(G)$.

EXAMPLE

E9: For the graph shown in Figure 6.4.1, the fundamental cutsets with respect to the spanning tree $T = \{e_1, e_2, e_4, e_7\}$ are

Branch e_1 $S_1 = \{e_1, e_3, e_5\}$,

Branch e_2 $S_2 = \{e_2, e_3, e_6, e_8\}$, Branch e_4 $S_4 = \{e_4, e_5, e_6, e_8),$

Branch e_7 $S_7 = \{e_7, e_8\}.$

It can be verified that the cut $= \{e_1, e_2, e_4, e_7, e_8\}$ containing the branches e_1, e_2, e_4, and e_7 is the ring sum of the fundamental cutsets S_1, S_2, S_4, and S_7. This illustrates Fact 20.

6.4.4 Relationship Between Circuit and Cutset Subspaces

By now it should be evident that circs and cuts are dual concepts in the sense that for each result that involves circuits or circs, there is a corresponding result involving cutsets or cuts. Facts 5 through 13 correspond to Facts 14 through 22. Spanning trees and cospanning trees provide the links between circs and cuts. This duality is further explored next.

Orthogonality of Circuit and Cutset Subspaces

DEFINITIONS

D24: The binary m-vector representing a circ is called a ***circuit vector***; the binary m-vector representing a cut is called a ***cut vector***; and the m-vector representing an incidence set is called an ***incidence vector***.

D25: Two subspaces W' and W'' of a vector space W are ***orthogonal*** to each other if the inner product (or dot product) of every vector in W' with every vector in W'' is zero. Note that the zero vector belongs to every subspace.

FACTS

F24: A circuit and a cutset of a connected graph have an even number of edges in common. Hence, a circ and a cut have an even number of edges in common.

F25: The inner product of a circuit vector and a cut vector over GF(2) is zero under the ring sum operation.

F26: A subgraph of a graph G belongs to the circuit subspace of the graph if and only it has an even number of edges in common with every subgraph in the cutset subspace of G. Equivalently, a vector is a circuit vector if and only if it is orthogonal to every cut vector.

F27: A subgraph of a graph G belongs to the cutset subspace of the graph if and only if it has an even number of edges in common with every subgraph in the circuit subspace of G. Equivalently, a vector is a cut vector if and only if it is orthogonal to every circuit vector.

F28: The circuit and cutset subspaces of a graph are orthogonal to each other.

Circ/Cut-Based Decomposition of Graphs and Subgraphs

DEFINITION

D26: Two orthogonal subspaces W' and W'' of a vector space W are ***orthogonal complements*** if every vector in W can be expressed as the ring sum of a vector of W' and a vector of W''. Note that the zero vector is the only vector that is in the intersection of the orthogonal complements W' and W''.

FACTS

F29: If the orthogonal subspaces W' and W'' of a vector space W are not orthogonal complements, then the dimension of their union is less than the dimension of the vector space W.

F30: [Ch71a] The circuit and the cutset subspaces of a graph are orthogonal complements if and only if the graph has an odd number of spanning forests.

F31: If the circuit and cutset subspaces of a graph are orthogonal complements, then every subgraph (including the graph itself) can be expressed as the ring sum of a circ and a cut.

F32: [Ch71b, WiMa71] Every graph can be represented as the ring sum of a circ and a cut of the graph. If the dimension of the intersection of the circuit and cutset subspaces of a graph is equal to k, then there are 2^k such representations.

EXAMPLES

E10: Consider the graph G_a in Figure 6.4.7. It can be verified that no nonempty subgraph of this graph is both a circ and a cut. So the cutset and circuit subspaces of G_a are orthogonal complements. Then the set of fundamental cutsets and fundamental circuits with respect to a spanning tree of G_a constitutes a basis of the vector space $\Psi(G)$. One such set with respect to the spanning tree formed by the edges e_1, e_2, e_3 and e_4 is as follows:

$$S_1 = (1\ 0\ 0\ 0\ 1\ 1\ 0)$$
$$S_2 = (0\ 1\ 0\ 0\ 1\ 1\ 0)$$
$$S_3 = (0\ 0\ 1\ 0\ 1\ 0\ 1)$$
$$S_4 = (0\ 0\ 0\ 1\ 0\ 0\ 1)$$
$$C_5 = (1\ 1\ 1\ 0\ 1\ 0\ 0)$$
$$C_6 = (1\ 1\ 0\ 0\ 0\ 1\ 0)$$
$$C_7 = (0\ 0\ 1\ 1\ 0\ 0\ 1)$$

It is easy to verify that every subgraph can be expressed as the ring sum of a circ and a cut, which illustrates Fact 31. For instance, the vector (0 0 1 1 0 1 1), which represents the induced subgraph on the edge subset $\{e_3, e_4, e_6, e_7\}$, can be expressed as:

$$(0\ 0\ 1\ 1\ 0\ 1\ 1) = S_1 \oplus S_2 \oplus C_6 \oplus C_7$$

$$= (1\ 1\ 0\ 0\ 0\ 0\ 0) \oplus (1\ 1\ 1\ 1\ 0\ 1\ 1)$$

where (1 1 0 0 0 0 0) represents a cut in G_a, and (1 1 1 1 0 1 1) represents a circ.

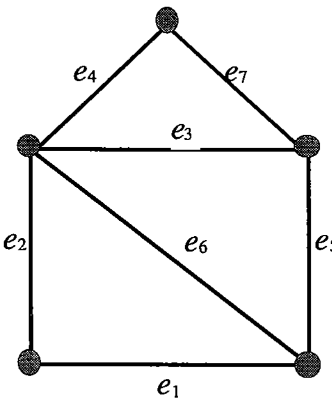


Figure 6.4.7 Graph G_a for illustration of Fact 31.

E11: Consider the graph G_b in Figure 6.4.8. In this graph the edges e_1, e_2, e_3, and e_5 constitute a circuit as well as a cut. Hence the circuit and cutset subspaces are not orthogonal complements. This means that there is a subgraph of G_b that cannot be expressed as the ring sum of a circ and a cut. However, according to Fact 32, such a decomposition is possible for G_b. This is verified as follows:

$$(1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1) = (1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0) \oplus (0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1)$$

where (1 1 0 1 0 0) represents the cut of edges e_1, e_2, and e_4, and (0 0 1 0 1 1) represents the circuit of edges e_3, e_5, and e_6 in G_b.

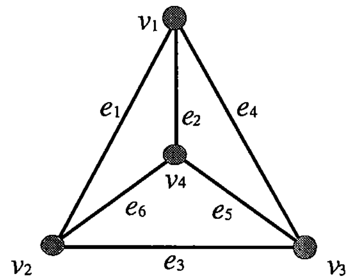


Figure 6.4.8 Graph G_b for illustration of Fact 32.

6.4.5 The Circuit and Cutset Spaces in a Directed Graph

In most engineering applications of graph theory, directed graphs are encountered. But, as we shall see next, the effect of orientation is minimal in so far as the results concerning circuits and cuts are concerned. Almost all the results presented earlier in this section have their equivalents in the directed case. In fact, we can view all the results on undirected graphs presented thus far as special cases of the results to be presented next.

TERMINOLOGY: A *circuit*, *cut*, or *spanning tree* in a directed graph G is a subset of edges that constitutes a circuit, cut, or spanning tree, respectively, in the underlying graph of G.

Circuit and Cut Vectors and Matrices

DEFINITIONS

D27: A circuit in a directed graph can be traversed in one of two directions, clockwise or counter-clockwise (relative to a plane drawing of the circuit). The traversal direction we choose is called the ***circuit orientation***.

D28: Let C be a circuit in a directed graph and $e = (v_i, v_j)$ an edge in C directed from v_i to v_j. Given an orientation of C, edge e is said to ***agree with the circuit orientation*** if the traversal of e specified by that orientation is from its tail v_i to its head v_j.

D29: A cut (V_a, V_b) in a directed graph can be traversed in one of two directions, from V_a to V_b or from V_b to V_a. The direction chosen is called the ***cut orientation***.

D30: Given an orientation of a cut in a directed graph, an edge $e = (v_i, v_j)$ in the cut is said to ***agree with the cut orientation*** if the traversal of e specified by that orientation is from v_i to v_j.

D31: Let G be a directed graph with edge-set $E = \{e_1, e_2, \ldots, e_m\}$, and let C be a circuit in G with a given orientation. The ***circuit vector*** representing C is the m-vector $(x_1, x_2, \ldots, x_m)$, where

$$x_i = \begin{cases} 1, & \text{if edge } e_i \text{ agrees with the orientation of } C \\ -1, & \text{if edge } e_i \text{ does not agree with the orientation of } C \\ 0, & \text{if edge } e_i \text{ is not in } C \end{cases}$$

D32: Let G be a directed graph with edge-set $E = \{e_1, e_2, \ldots, e_m\}$, and let S be a cut in G with a given orientation. The ***cut vector*** representing S is the m-vector $(x_1, x_2, \ldots, x_m)$, where

$$x_i = \begin{cases} 1, & \text{if edge } e_i \text{ agrees with the orientation of } S \\ -1, & \text{if edge } e_i \text{ does not agree with the orientation of } S \\ 0, & \text{if edge } e_i \text{ is not in } S \end{cases}$$

D33: Let G be a directed graph with edge-set $E = \{e_1, e_2, \ldots, e_m\}$. Let $C_1, C_2, \ldots, C_t$ and $S_1, S_2, \ldots, S_r$ be the circuits and cuts of G, respectively, each with a given traversal orientation. The ***circuit matrix*** of G is the $t \times m$ matrix whose ith row is the circuit vector representing circuit C_i. The ***cut matrix*** of G is the $r \times m$ matrix whose ith row is the cut vector representing cut S_i.

The Fundamental Circuit, Fundamental Cutset, and Incidence Matrices

Next, we define two special matrices corresponding to the fundamental circuits and cutsets relative to a given spanning tree in a directed graph and a third matrix corresponding to the incidence vectors of the vertices.

REMARK

R9: The definitions of these three matrices depend on how the associated circuits and cuts are oriented. The orientations of each fundamental circuit and each fundamental cut are usually chosen to agree with the defining chord and branch, respectively, and we adopt that convention here. Also, for the cut consisting of the set of edges incident on a vertex v (i.e., the incidence set of v), we assume that the orientation is away from vertex v. Accordingly, the incidence vector of vertex v is given by $(x_1, x_2, \ldots, x_m)$, where

$$x_i = \begin{cases} 1, & \text{if edge } e_i \text{ is directed } \textit{from } v \quad (v \text{ is the tail of edge } e_i) \\ -1, & \text{if edge } e_i \text{ is directed } \textit{to } v \quad (v \text{ is the head of edge } e_i) \\ 0, & \text{if edge } e_i \text{ is not incident on } v \end{cases}$$

DEFINITIONS

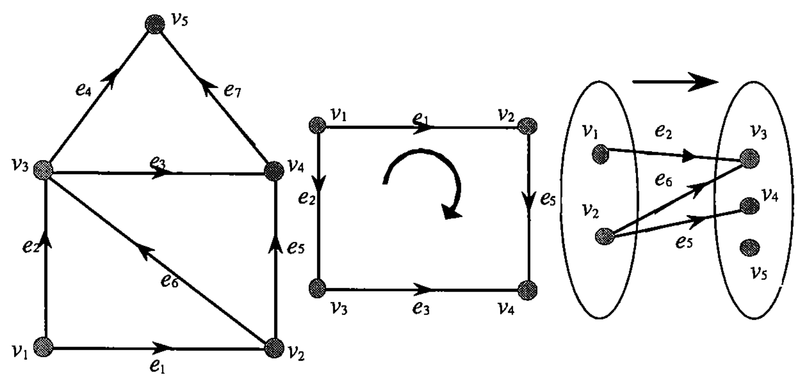
D34: Let T be a spanning tree of a connected directed graph. The ***fundamental circuit matrix*** of the graph with respect to T, denoted by $\boldsymbol{B}_f$, is the $(m - n + 1)$-rowed submatrix of the circuit matrix whose rows are the fundamental circuit vectors. Similarly, the ***fundamental cutset*** matrix with respect to T, denoted by Q_f, is the $(n-1)$-rowed submatrix of the cut matrix whose rows are the fundamental cutset vectors.

D35: The ***incidence matrix*** of a given directed graph, denoted A_c, is the n-rowed submatrix of the cut matrix whose rows are the incidence vectors of the directed graph. The submatrix of the incidence matrix containing any $n - 1$ of the incidence vectors is called a ***reduced incidence matrix*** and is denoted by A.

D36: A matrix of real numbers is ***unimodular*** if the determinant of every square submatrix of the matrix is equal to 1, -1 or 0.

EXAMPLE

E12: Consider the directed graph of Figure 6.4.9(a). A circuit and a cut with orientations are shown in Figure 6.4.9(b) and Figure 6.4.9(c), respectively. The corresponding circuit and cut vectors are (1 -1 -1 0 1 0 0) and (0 1 0 0 1 1 0), respectively.



(a) A directed graph. (b) A circuit with orientation (c) A cut with orientation

Figure 6.4.9 **A directed graph, a circuit, and a cut with orientations**

E13: Consider the spanning tree T of the graph of Figure 6.4.9(a) consisting of the edges e_1, e_2, e_3, and e_4. The fundamental circuit and the fundamental cutset matrices with respect to T, and the incidence matrix of this graph with the column i in each matrix corresponding to edge e_i are:

Fundamental Circuit Matrix:

$$\begin{array}{r} \text{Chord } e_5 \\ \text{Chord } e_6 \\ \text{Chord } e_7 \end{array} \begin{pmatrix} 1 & -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

Fundamental Cutset Matrix:

$$\begin{array}{r} \text{Branch } e_1 \\ \text{Branch } e_2 \\ \text{Branch } e_3 \\ \text{Branch } e_4 \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Incidence Matrix:

$$\begin{array}{r} \text{Node } v_1 \\ \text{Node } v_2 \\ \text{Node } v_3 \\ \text{Node } v_4 \\ \text{Node } v_5 \end{array} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \end{pmatrix}$$

Orthogonality and the Matrix Tree Theorem

TERMINOLOGY: A directed edge which is in both a circuit and a cut is said to have the ***same relative orientation*** with respect to the circuit and the cut if the edge either agrees or disagrees with the assigned orientations of both the circuit and the cut.

FACTS

F33: A circuit and a cut in a connected graph have an even number of common edges. If a circuit and a cut have $2k$ common edges, then these edges can be partitioned into two sets, each of size k, such that each edge in one set has the same relative orientation with respect to the circuit and the cut, and that each edge in the other set agrees with one of the two assigned orientations (circuit or cut) and disagrees with the other assigned orientation.

NOTATION: Let G be a directed graph and suppose that each of the circuits and cuts has been given an orientation.
(a) The collection of all circuit vectors of G and their linear combinations over the real field is denoted by $\hat{C}(G)$.
(b) The collection of all cut vectors of G and their linear combinations over the real field is denoted by $\lambda(G)$.

F34: In a directed graph every circuit vector is orthogonal to every cut vector over the real field.

F35: In a connected directed graph, every circuit vector can be expressed as a linear combination of fundamental circuit vectors with respect to a spanning tree of the graph. The coefficients in the linear combination are 1 or -1. Similarly, every cut vector in a connected directed graph can be expressed as a linear combination of fundamental cutset vectors with respect to a spanning tree of the graph. The coefficients in the linear combination are 1 or -1.

F36: In a directed graph G, $\hat{C}(G)$ and $\lambda(G)$ are vector spaces over the real field and are orthogonal to each other. $\hat{C}(G)$ and $\lambda(G)$ are called the ***circuit space*** and the ***cutset space***, respectively.

F37: The fundamental circuit vectors and the fundamental cutset vectors with respect to a spanning tree of a connected directed graph G form a basis of the circuit space and a basis of the cutset space, respectively. The dimension of the circuit space, is equal to $m-n+p$, the nullity of G, and the dimension of the cutset space is equal to $n-p$, the rank of G, where p is the number of components of G.

F38: Any set of $n-1$ incidence vectors of a connected directed graph forms a basis of the cutset space of the graph.

F39: The fundamental cutset and the fundamental circuit matrices of a connected directed graph are unimodular.

F40: Consider a spanning tree T of a connected directed graph G with branches $b_1, b_2, \ldots\ldots, b_{n-1}$ and chords $c_1, c_2, c_3, \ldots, c_{m-n+1}$. Suppose that the edges of G are labeled so that $e_1, e_2, \ldots, e_m = b_1, b_2, \ldots, b_{n-1}, c_1, c_2, \ldots, c_{m-n+1}$, respectively. Then the fundamental circuit matrix B_f has the form $B_f = [B_{ft}|U_{m-n+1}]$, where U_{m-n+1} is the identity matrix of size $m-n+1$ and B_{ft} is the submatrix of B_f consisting of the columns corresponding to the branches $b_1, b_2, \ldots, b_{n-1}$ of T. Similarly, the fundamental cutset matrix Q_f has the form $Q_f = [U_{n-1}|Q_{fc}]$, where U_{n-1} is the identity matrix of size $n-1$ and Q_{fc} is the submatrix of Q_f consisting of the columns corresponding to the chords $c_1, c_2, \ldots, c_{m-n+1}$ of T. Moreover, $Q_{fc} = -B^t_{ft}$.

F41: The columns of the cut matrix of a connected directed graph G are linearly independent if and only if they correspond to the branches of a spanning tree. Similarly, the columns of the circuit matrix are linearly independent if and only if they correspond to the chords of a cospanning tree.

F42: (*Matrix Tree Theorem*) For a connected directed graph, each cofactor of the matrix $A_c A_c^t$ equals the number of spanning trees of the graph.

EXAMPLE

E14: The matrices in Example 13 illustrate Facts 33 through 41.

REMARKS

R10: By simply replacing -1 by 1 in all the matrices defined for directed graphs, we get the corresponding matrices for undirected graphs.

R11: The rank and the nullity of the cut matrix of a connected graph are $(n-1)$ and $(m-n+1)$, respectively. This motivated the definitions of the rank and nullity of a graph (See Definition 13).

R12: The matrix $A_c A_c^t$ is called the *degree matrix* of the graph. It can be verified that the diagonal entry (i,i) of the degree matrix is equal to the degree of vertex v_i and the off-diagonal entry (i,j) is equal to the negative of the number of edges connecting vertex v_i and vertex v_j (regardless of the orientations of these edges). A proof of Fact 42 may be found in [ThSw92]. A weighted version of the degree matrix plays an important role in electrical circuit analysis [SwTh81].

Minty's Painting Theorem

TERMINOLOGY: Two directed edges in a circuit or cutset are said to have the ***same direction*** (relative to that circuit or cutset) if both edges agree with the same orientation of that circuit or cutset.

DEFINITIONS

D37: A ***directed circuit*** is a circuit whose edges all have the same direction relative to it.

D38: A ***directed cutset*** is a cutset whose edges all have the same direction relative to it.

D39: A ***painting*** of a directed graph G is a partitioning of the edges of the graph intro three sets R, Y, and B and the distinguishing of one element of the set Y. We can visualize this as coloring of the edges of G with three colors, each edge being painted red, yellow, or blue, and exactly one yellow edge being colored dark yellow.

FACTS

F43: (*Painting Theorem*) [Mi66] Let G be a directed graph. For any painting of the edges of G, exactly one of the following holds:

1. There exists a circuit containing the dark yellow edge but no blue edges, in which all the yellow edges have the same direction as the dark yellow edge.
2. There exists a cutset containing the dark yellow edge but no red edges, in which all the yellow edges have the same direction as the dark yellow edge.

F44: Each edge of a directed graph is in a directed circuit or in a directed cutset, but no edge belongs to both.

REMARK

R13: Minty's painting theorem (also known as the ***arc coloring lemma***) has profound applications in electrical circuit theory. This theorem is also true for orientable matroids (See [ThSw92]). Fact 44 is a corollary of Fact 43. Other related works by Minty of considerable significance in electrical circuit theory are [Mi60, Mi61]. Some applications of the arc coloring lemma to problems in electrical circuit theory may be found in [VaCh80, ChGr76, Wo70].

6.4.6 Two Circ/Cut-Based Tripartitions of a Graph

In Section 6.4.4 we presented a result on the decomposition of a graph into a circ and a cut. But such circs and cuts may not be disjoint and hence they may not form a partition of the edge set of the graph. We now present two ways to partition a graph. These partitions are both tripartitions and are again based on circs and cuts.

Bicycle-Based Tripartition

DEFINITION

D40: A subgraph that is in the intersection of the circuit and cutset subspaces of an undirected graph is called a ***bicycle***. That is, a bicycle is a circ as well as a cut.

EXAMPLE

E15: The edges e_1, e_2, e_3, and e_5 in the graph of Figure 6.4.8 form both a cut and a circuit.

FACT

F45: [RoRe78] Any edge e of a graph G is of one of the following types:

1. e is in a circ which becomes a cut when e is removed from it.
2. e is in a cut which becomes a circ when e is removed from it.
3. e is in a bicycle.

TERMINOLOGY: The partition of the edges defined by Fact 45 is called the ***bicycle-based tripartition***.

REMARK

R14: Roentiehl and Read [RoRe78] have proved several interesting results relating to circuits and cuts and their relationship. A proof of Fact 45 may also be found in [Pa94].

A Tripartition Based on Maximally Distant Spanning Trees

DEFINITIONS

D41: The ***tree distance***, $d(T_1, T_2)$, between any two spanning trees T_1 and T_2 is defined as $d(T_1, T_2) = |E(T_1) - E(T_2)| = |E(T_2) - E(T_1)|$.

D42: Two spanning trees T_1 and T_2 are ***maximally distant*** if $d(T_1, T_2) \geq d(T_i, T_j)$ for every pair of spanning trees T_i and T_j.

NOTATION: The maximum distance between any two spanning trees of a connected graph is denoted by d_m.

D43: Given a pair of maximally distant spanning trees T_1 and T_2 of a connected graph G. Suppose c is a common chord of T_1 and T_2. The ***k-subgraph*** G_c of G with respect to c is the edge-induced subgraph constructed as follows:

1. Let L_1 be the set of all the edges in the fundamental circuit with respect to T_1 defined by c.
2. Let L_2 be the union of the sets of edges in all the fundamental circuits with respect to T_2 defined by every edge in L_1.
3. Repeating the above, we can obtain a sequence of sets of edges L_1, $L_2, \ldots$ until we arrive at a set $L_{k+1} = L_k$. Then the induced subgraph on the edge set L_k is called *the k-subgraph* G_c with respect to c.

D44: The ***k-subgraph*** G_b with respect to a common branch b can be constructed in a dual manner as in Definition 43.

D45: The ***principal subgraph*** G_1 with respect to the common chords (of a pair of maximally distant spanning trees T_1 and T_2) is the union of the k-subgraphs with respect to all the common chords. The ***principal subgraph*** G_2 with respect to the common branches is the union of the k-subgraphs with respect to all the common branches.

FACTS

F46: [KiKa69] Let T_1 and T_2 form a pair of maximally distant spanning trees of a connected graph G.

1. The fundamental circuit of G with respect to T_1 or T_2 defined by a common chord of T_1 and T_2 contains no common branches of these spanning trees.
2. The fundamental cutset of G with respect to T_1 or T_2 defined by a common branch of T_1 and T_2 contains no common chords of these spanning trees.

F47: [KiKa69] Consider a graph $G = (V, E)$. Let E_1 and E_2 denote the edge-sets of the principal subgraphs G_1 and G_2, respectively, and let $E_0 = E(G) - (E_1 \cup E_2)$. Then E_0, E_1, and E_2 form a partition of the edge-set $E(G)$. The partition (E_0, E_1, E_2) is called the ***principal partition*** of G and is independent of the maximally distant trees used to construct it.

EXAMPLES

E16: It can be verified that $T_1 = \{e_2, e_3, e_4, e_7\}$ and $T_2 = \{e_1, e_3, e_5, e_6\}$ are a pair of maximally distant spanning trees for the graph in Figure 6.4.10 and that the associated principal partition is: $E_1 = \{e_6, e_7, e_8\}$, $E_2 = \{e_1, e_2, e_3\}$, and $E_0 = \{e_4, e_5\}$.

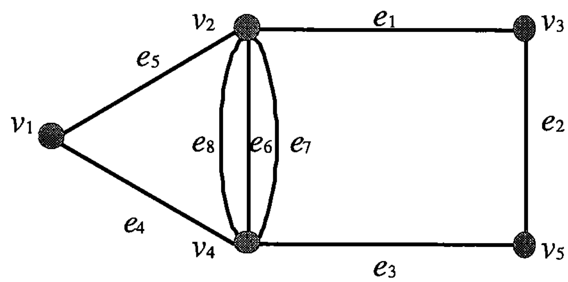


Figure 6.4.10

REMARKS

R15: In electrical circuit analysis one is interested in solving for all the current and the voltage variables. The circuit method of analysis (also known as the loop analysis) requires solving for only $m-n+1$ independent current variables. The remaining current variables and all the voltage variables can then be determined using these $m - n + 1$ independent current variables. The cutset method of analysis requires solving for only $n - 1$ independent voltage variables. A question that intrigued circuit theorists for a long time was whether one could use a hybrid method of analysis involving some current variables and some voltage variables and reduce the size of the system of equations to be solved to less than both $n - 1$ and $m - n + 1$, the rank and nullity of the graph of the circuit. Ohtsuki, Ishizaki and Watanabe [OhIsWa70] studied this problem and showed that d_m, the maximum distance between any two spanning trees of the graph of the circuit is, in fact, the minimum number of variables required in the hybrid method of analysis. They also showed that the variables can be determined using the principal partition of the graph. The works by Kishi and Kajitani [KiKa69] on principal partition and by Ohtsuki, Ishizaki and Watanabe [OhIsWa70] on the hybrid method of analysis are considered landmark results in electrical circuit theory. Swamy and Thulasiraman [SwTh81] give a detailed exposition of the principal partition concept and the hybrid and other methods of circuit analysis.

R16: Lin [Li76] presented an algorithm for computing the principal partition of a graph. Bruno and Weinberg [BrWe71] extended the concept of principal partition to matroids.

6.4.7 Realization of Circuit and Cutset Spaces

In the application of graph theory to the electrical circuit synthesis problem, one encounters a certain matrix of integers modulo 2 and seeks to determine if this matrix is the cutset or the circuit matrix of an undirected graph. The complete solution to this problem was given by Tutte [Tu59]. Cederbaum [Ce58] and Gould [Go58] considered this problem before Tutte provided the solution. We now present the main result on the necessary and sufficient conditions for the realizability of a matrix of integers modulo 2 as the circuit or the cutset matrix of an undirected graph. Related results leading to this main result are also presented. Seshu and Reed [SeRe61] discuss these results in considerable detail, except for the proof of the sufficiency of Tutte's realizability condition.

DEFINITIONS

D46: The graphs in Figure 6.4.11 are called **Kuratowski graphs**.

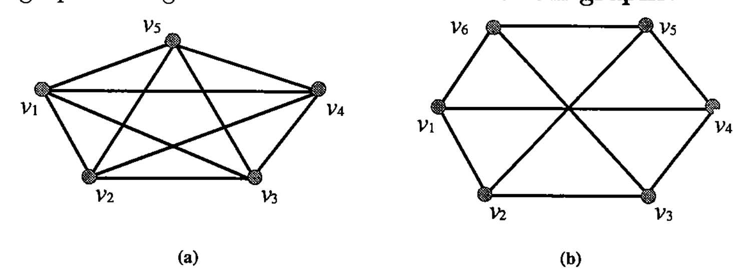


Figure 6.4.11 The two Kuratowski graphs

D47: A matrix F of the form $F = [F' \mid U]$, where U is the identity matrix, is said to be in ***normal form***.

D48: A matrix F of real integers in normal form is a ***regular matrix*** if for every linear combination X of the rows of M with coefficients -1, 1, and 0 we have the following:

1. The elements of X are 1, -1, and 0, or
2. There exists another such linear combination Y (with coefficients 1, -1, and 0) that has 1 and -1 for nonzero elements and these are at a (not necessarily proper) subset of the positions in which X has nonzero elements.

D49: A matrix of integers mod 2 is ***regular*** if the replacement of a suitable set of 1's by -1's makes it regular.

FACTS

F48: For a connected directed graph G, the fundamental cutset and fundamental circuit matrices with respect to a spanning tree T of G and the reduced incidence matrix A of G are all regular matrices.

F49: A regular matrix in normal form is unimodular.

F50: Given a regular matrix F of integers 1, -1, and 0 in normal form, the replacement of -1's by 1's will leave the ranks of the submatrices unaltered (where the rank of the derived matrix is with respect to modulo 2 arithmetic).

F51: A matrix F of integers mod 2 is regular if and only if no normal form of F contains either the matrix N_0 or its transpose, where

$$N_0 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

F52: A matrix F of integers mod 2 is realizable as the cutset matrix of an undirected graph if and only if it is regular and no normal form of F contains the circuit matrix of either of the two Kuratowski graphs shown in Figure 6.4.11.

F53: A matrix F of integers mod 2 in normal form is realizable as the circuit matrix of an undirected graph if and only if it is regular and no normal of F contains the cutset matrix of either of the two Kuratowski graphs shown in Figure 6.4.11.

REMARKS

R17: Mayeda [Ma70] gave an alternate proof of Tutte's realizability condition, which is shorter than Tutte's original proof that is 27 pages long.

R18: Early works on algorithms for constructing graphs having specified circuit or cutset matrices are in [Tu60, Tu64]. Bapeswara Rao [Ba70] defined the tree- path matrix of an undirected graph which is essentially the non-unit submatrix of the fundamental circuit matrix and presented an algorithm for constructing a graph with a prescribed tree path-matrix. This is also an algorithmic solution to the cutset and the circuit matrix realization problems. A detailed presentation of Bapeswara Rao's algorithm is given in [SwTh81].

R19: The circuit and the cutset matrix realization problems arise in the design of multi-port resistance networks. It was in the context of this application that Cederbaum [Ce58, Ce59] encountered the realization problem. Interestingly, Bapeswara Rao [Ba70] and Boesch and Youla [BoYo65] presented circuit-theoretic approaches to the realization of a matrix as the cutset or circuit matrix of a directed graph. Details of Bapeswara Rao's algorithm based on this approach may also be found in [SwTh81].

Whitney and Kuratowski

We believe that it is appropriate to conclude this section with a reference to two classic works by Whitney [Wh33] and Kuratowski [Ku30] relating to duality. While the results of this section bring out the duality between circuits and cutsets, Whitney introduced the concept of duality between graphs. His original definition was an algebraic one (see also [ThSw92]) relating the nullity and rank of certain corresponding subgraphs of dual graphs. Definition 50 is an equivalent one.

DEFINITION

D50: A graph G_2 is a ***dual*** of a graph G_1 if there is a one-to-one correspondence between their edge-sets such that a set of edges in G_2 is a circuit vector of G_2 if and only if the corresponding set of edges in G_1 is a cutset vector of G_1.

FACTS

F54: It follows from the duality between circuits and cutsets that if G_2 is a dual of G_1, then G_1 is a dual of G_2.

F55: [Wh33] A graph has a dual if and only if it is planar.

F56: In another classic work, Kuratowski [Ku30] proved that a graph is planar if and only if it does not contain a subdivision of a Kuratowski graph.

REMARK

R20: See [We01] for a proof of Kuratowski's theorem. It is quite interesting to see the role of the Kuratoswki graphs in Tutte's realizability conditions for the cutset and the circuit matrix realization problems.

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GLOSSARY

bicycle- of a graph: a subgraph which is both a circ and a cut.

binary vector- representing a subset of edges E' in an undirected graph: a row vector, the ith component of which is 1 if the ith edge of the graph is in E' and is 0, otherwise.

branch – of a spanning tree: an edge of the spanning tree.

chord – of a cospanning tree: an edge of the cospanning tree.

circ – of a graph: a circuit or union of edge-disjoint circuits of the graph.

circuit matrix – of a graph (directed or undirected): the matrix in which each row is a circuit vector and its number of rows is equal to the number of circs in the graph.

circuit subspace – of an undirected graph G: the set of all circs of the graph and is denoted by $\hat{C}(G)$.

circuit space – of a directed graph G: the set of all circuit vectors and their linear combinations over the real field, and is denoted by $\hat{C}(G)$.

circuit vector – of an undirected graph: the binary m-vector representing a circ of the graph.

circuit vector – of a directed graph: the m-vector representing a circ of the graph. The signs of the elements in the vector depend on the orientation assigned to each of the circuits in the circ.

complement of $G' = (V', E')$ - in $G = (V, E)$: graph $G'' = (V, E - E')$.

component – of a graph: a maximal connected subgraph of the graph.

connected graph: graph in which there is a path between every pair of vertices.

cospanning tree – of a graph G with respect to a spanning tree T: complement of T in G.

cut $\langle V_1, V_2 \rangle$ – of a graph $G = (V, E)$: the set of edges with one end vertex in V_1 and the other in $V_2 = V - V_1$.

cut matrix – of a graph (directed or undirected): the matrix in which each row is a cut vector and its number of rows is equal to the number of cuts in the graph.

cut vector – of an undirected graph: the binary m-vector representing a cut of the graph.

cut vector – of a directed graph: the m-vector representing a cut of the graph . The signs of the elements in the vector depend on the cut orientation.

cutset – of a connected graph G: a cut whose removal results in a graph with exactly two components.

cutset subspace - of a graph G: the set of all cuts of the graph and is denoted by $\lambda(G)$.

cutset space - of a directed graph G: the set of all cut vectors of G and their linear combinations over the real field, and is denoted by λ (G).

degree – of a vertex: the number of edges incident on a vertex.

difference of two sets S_1 and S_2: $S_1 - S_2$ is the set of all elements in S_1 but not in S_2.

directed circuit – of a directed graph: a circuit in which all the edges are oriented in the same direction.

directed cutset – of a directed graph: a cut in which all the edges are oriented in the same direction.

dual - of a graph: see Definition 50.

edge-induced subgraph on E'- of a graph $G = (V, E)$: graph $G' = (V', E')$ with $E' \subseteq E$ and V' consisting of only those vertices which are endpoints of the edges in E'.

empty graph: a graph with no edges.

even graph: a graph in which the degree of every vertex is even.

fundamental circuit – of a graph with respect to a chord: the unique circuit produced by adding a chord to a spanning tree of the graph. C_i will denote the fundamental circuit with respect to the chord e_i.

fundamental cutset – of a graph with respect to a branch: the unique cutset $< V_1, V_2 >$, where V_1 and V_2 are the sets of vertices of the two trees that result when the branch is removed from the spanning tree. S_i will denote the fundamental cutset with respect to the branch e_i.

fundamental circuit matrix – of a connected graph with respect to a spanning tree: the $(m - n + 1)$-rowed submatrix of the circuit matrix in which each row is a fundamental circuit vector with respect to the spanning tree, and will be denoted by $\boldsymbol{B}_f$. In the case of directed graphs, the orientation of the fundamental circuit is chosen to agree with the orientation of the chord defining the fundamental circuit.

fundamental cutset matrix – of a connected graph with respect to a spanning tree: the $(n - 1)$-rowed submatrix of the cut matrix in which each row is a fundamental cutset vector with respect to the spanning tree, and will be denoted by Q_f. In the case of directed graphs, the orientation of a fundamental cutset is chosen to agree with the orientation of the branch defining the fundamental cutset.

incidence matrix – of a graph: the n-rowed submatrix of the cut matrix in which each row is an incidence vector.

incidence set – of a graph: the set of edges incident on a vertex of the graph.

incidence vector – of an undirected graph: the binary cut vector representing the set of edges incident on a vertex of the graph.

incidence vector – of a directed graph: the cut vector representing the set of edges incident on a vertex of the graph with the orientation of the cut chosen to be away from the vertex.

incident on: edge $e = (v_i, v_j\ )$ is incident on vertices v_i and v_j.

isolated vertex: a vertex with degree zero.

Kuratowski graph: either of the two graphs in Figure 6.4.11.

maximally distant trees: two spanning trees T_1 and T_2 are maximally distant if $d(T_1, T_2) \geq d(T_i, T_j)$ for every pair of spanning trees T_i and T_j.

normal form – of a matrix: a matrix of the form $[F' \mid U]$, where U is the identity matrix.

null graph: a graph with no vertices and hence no edges and is denoted by $\emptyset$.

nullity – of a graph G having n vertices, m edges and p components: nullity is equal to $m - n + p$ and is denoted $\mu(G)$.

orthogonal complements – of a vector space: two subspaces of a vector space are orthogonal complements if the zero vector is the only vector in the intersection of the two subspaces.

orthogonal subspaces – of a vector space: two subspaces of a vector space are orthogonal if the inner product of every vector in one subspace with every vector in the other subspace is equal to zero.

orientation – of a circuit in a directed graph: the direction we choose to traverse the circuit.

orientation of cut $\langle V_1, V_2\rangle$ - in a directed graph: the direction, from V_1 to V_2 or from V_2 to $V_{1.}$, we choose for the cut.

principal subgraph G_1 – of a graph G : the union of principal subgraphs with respect to all common chords of a pair of maximally distant spanning trees of G.

principal subgraph G_2 – of a graph G : the union of principal subgraphs with respect to all common branches of a pair of maximally distant spanning trees of G.

painting – of a graph: partitioning of the edges of the graph intro three sets R (red), Y (yellow), and B (blue) and the distinguishing of one element of the set Y.

rank – of a graph G having n vertices and p components: rank is equal to $n - p$ and is denoted by $\rho(G)$.

reduced incidence matrix – of a graph: the submatrix of the incidence matrix containing any $(n-1)$ incidence vectors.

regular matrix: see Definitions 48 and 49.

ring sum - of two sets E_1 and E_2: the set consisting of only elements which belong to E_1 or to E_2 but not to both E_1 and E_2, and is denoted by $E_1 \oplus E_2$.

ring sum - of two vectors $(x_1, x_2, x_3, \ldots..x_i, \ldots..x_m)$ and $(y_1, y_2, y_3, \ldots..y_i, \ldots..y_m)$: the vector Z $= (z_1, z_2, z_3, \ldots..z_i, \ldots..z_m)$, where $z_i = x_i \otimes y_i$ and $\otimes$ is the logical exclusive-or operation ($1 \otimes 0 = 1$, $0 \otimes 1 = 1$, $0 \otimes 0 = 0$, and $1 \otimes 1 = 0$).

spanning forest - of G having p components: collection of p spanning trees, one for each component of G.

spanning tree – of a connected graph: a tree of the graph which contains all the vertices of the graph.

symmetric difference – of two sets E_1 and E_2: the set consisting of only elements which belong to E_1 or to E_2 but not to both E_1 and E_2, and is denoted by $E_1 \oplus E_2$.

tree – of a graph: a connected subgraph of the graph containing no circuits.

trivial graph: graph with a single vertex and no edge.

unimodular matrix: a matrix of real numbers, the determinant of every square submatrix of which is equal to 1, -1, or 0.

vector space – of a graph G: the set of all subsets of edges of G and is denoted by ψ(G); also the set of all vectors representing the subsets of edges of G.

vertex-induced subgraph on V' – of a graph $G = (V, E)$: graph $G' = (V', E')$ with $V' \subseteq V$ and E' consisting of only those edges whose endpoints are in V'.

Symbols

$\oplus$: ring sum or symmetric difference.

$\otimes$: the logical exclusive-or operation.

$\boldsymbol{d}_m$: the maximum distance between any two spanning trees.

ψ **(G)**: vector space of undirected graph G.

$\boldsymbol{C}$ **(G)**: circuit space of graph G.

$\lambda(G)$: cutset space of graph G.

$\rho(G)$: rank of graph G.

μ (G): nullity of graph G.

$\boldsymbol{B}_f$: fundamental circuit matrix.

$\boldsymbol{Q}_f$: fundamental cutset matrix.

A_c: incidence matrix.

$\boldsymbol{A}$: reduced incidence matrix.

$\boldsymbol{C}_i$: fundmetrnal circuit with respect to chord e_i

$\boldsymbol{S}_i$: fundamental cutset with respect to branch b_i.

ch	Working Section Title	Authors		Affiliation
1	INTRODUCTION			
	Basic definitions	GY (Gross and Yellen)		
	Common graph families	Beineke	Lowell	Purdue Ft Wayne
	Paths, cycles, and trees	GY		
	Subgraphs and graph operations	GY		
	History of graph theory	Wilson	Robin	Open Univ (UK)
2	EPRESENTATIONS OF GRAPHS			
	Computer repre (incl matrix)	Aho	Alfred	Columbia
	Graph isomorphism	Goldberg	Mark	RPI
	Graph reconstruction	Lauri	Josef	U Malta (MT)
	Recursive graph representation	Borie	Richard	U. of Alabama
		Parker	R.G.	Ga Tech
		Tovey	Craig	Ga Tech
3	DIRECTED GRAPHS			
	Digraph properties	Yellen	Jay	Rollins College
	DAG's, include rooted trees, topsort	Maurer	Steve	Swarthmore
	Tournaments	Reid	Brooks	Cal St, San Marcos
4	CONNECTIVITY and TRAVERSABILITY			
	Connectivity	Fabrega	Josep	U Poli de Catalunya (S
		Fiol	Miguel	U Poli de Catalunya (S
	Eulerian graphs	Fleischner	Herbert	Austrian Acad of Sci
	Chinese postman problem	Parker	Gary	Ga Tech
	Hamiltonian graphs	Gould	Ronald	Emory
	TSP and other routing problems	Gutin	Gregory	Royal Holloway (UK)
5	LORINGS and RELATED TOPICS			
	Graph coloring	Tuza	Zsolt	Hung. Acad. of Sci. (H
	Independence and cliques	Gutin	Gregory	Royal Holloway (UK)
	Factors and factorization	Plummer	Michael	Vanderbilt
	Perfect graphs	Tucker	Alan	SUNY Stony Brook
	Applications to timetabling and	DeWerra	Dominique	Ecole Poly Laus. (SW
	scheduling	Burke	Edmund	U. Nottingham (UK)
		Kingston	Jeff	U. Sydney (AU)
6	ALGEBRAIC GRAPH THEORY			
	Automorphisms	Watkins	Mark	Syracuse U.
	Cayley graphs	Alspach	Brian	U Regina-Sask (CA)
	Enumeration	Stockmeyer	Paul	Wm & Mary
	Graphs and vector spaces	Thulasiramai	K	U. Oklahoma
	Spectral graph theory	Doob	Michael	
	Matroidal methods in graph theory	Oxley	James	Louisiana State Univ
7	OPOLOGICAL GRAPH THEORY			
	Graphs on surfaces	Pisanski	Tomaz	U Ljubljana (SL)
		Potocnik	Primoz	U Ljubljana (SL)
	Voltage graphs	Gross	Jonathan	Columbia U
	Minimum and maximum genus	Chen	Jianer	Texas A&M
	Genus distributions	Gross	Jonathan	Columbia U
	Genus of a group	Tucker	Thomas	Colgate U
	Map theory	Vince	Andrew	U Fla
	Representativity	Archdeacon	Dan	U. Vermont
	Triangulations	Negami	Seiya	Yokohama Natl U (JP
	Graphs and finite geometries	White	Arthur	W. Mich U
8	ANALYTIC GRAPH THEORY			

	Extremal graph theory	Bollobas	Bela	Cambridge U (UK)
	Random graphs and asymptotics	Wormald	Nick	U. Melbourne (AU)
	Ramsey graph theory	Faudree	Ralph	U Memphis
	Probabilistic methods	Frieze	Alan	CMU
9	GRAPHICAL MEASUREMENT			
	Distance in graphs	Chartrand	Gary	W. Mich U.
		Zhang	Ping	W. Mich U.
	Domination	Haynes	Teresa	E. Tennessee State
		Henning	Michael	U. of Natal (S.Af)
	Tolerance graphs	McMorris	F.R.	Ill Inst of Tech
	Bandwidth	Brigham	R.C.	U. Central Fla
10	COMPUTER SCIENCE			
	Graph theory in computer science	Dewdney	Kee	U. Waterloo (CA)
	Dynamic graph algorithms	Italiano	Giuseppe	U Rome (IT)
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		Demetrescu	Camil	
	Graph searching	Gabow	Hal	U Colorado
	Algorithms on recursively defined grap	Borie	Richard	U. of Alabama
		Parker	R.G.	Ga Tech
		Tovey	Craig	Ga Tech
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		Liotta	Giuseppe	U. Perugia (IT)
11	NETWORKS and FLOWS			
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	Minimum cost flows	Fleischer	Lisa	CMU
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		Mirchandani	Prakash	U. Pitt