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Synthesis of the K-matrix of $(n + 3)$-node Resistive n-port Networks

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With 3 Figures

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1. Introduction

In this paper we establish a simple set of necessary and sufficient conditions for the synthesis of the K-matrix of an $(n + 3)$-node resistive n-port network [1, 2]. The approach given in [2] is followed. This approach is discussed briefly in this section.

Consider a resistive n-port network N. Let the port configuration T of N be in p parts $T_1, T_2, \ldots, T_p$. It may be assumed, without any loss of generality, that each T_i is a lagrangian tree. The set of vertices of T_i will be denoted by $i_0, i_1, i_2, \ldots, i_{n_i}$. The m-th port of T_i will be denoted by $P_{i(m)}$. i_0 and i_m are the negative and positive reference terminals of $P_{i(m)}$. T_i and the polarities of $P_{i(m)}$ are as shown in Fig. 1.

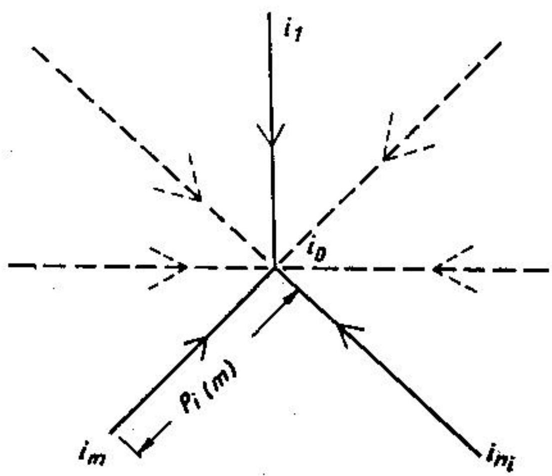

Fig. 1. T_i and the polarities of $P_{i(m)}$

Denoting by $g_{i_k j_m}$ the conductance of N connecting vertices i_k and j_m we then define $S_{i_k j}$ and S_{ij} as follows:

$$S_{i_k j} = \sum_{m=0}^{n_j} g_{i_k j_m}, \; j \neq i; \quad \text{and} \quad S_{ij} = \sum_{k=0}^{n_i} S_{i_k j} = \sum_{m=0}^{n_j} S_{j_m i}, \quad j \neq i. \tag{1}$$

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The network of departure and the padding network of N will be denoted respectively by N_d and N_p [3]. The subscripts "d" and "p" will be used to refer to the quantities of N_d and N_p. It can be shown that $(S_{ij})_p = S_{ij}$ and $(S_{i_k j})_p = S_{i_k j}$.

Let the potential factor matrix $K = [k_{ij}]$ of N be partitioned as

$$K = \begin{bmatrix} K_{11}\, K_{12} \cdots K_{1p} \\ K_{21}\, K_{22} \cdots K_{2p} \\ \cdots\cdots\cdots\cdots\cdots \\ K_{p1}\, K_{p2} \cdots K_{pp} \end{bmatrix}$$

where

i) each submatrix K_{ii}, $i = 1, \ldots, p$ has entries comprising 0's and 1's only, and is uniquely determined by T_i,

ii) each entry in K_{ij}, $j \neq i$ is less than or equal to unity and greater than or equal to zero.

Let $k_{i(k),j}$ denote the voltage of the set of ports in T_j with respect to i_0 when port $P_{i(k)}$ is excited with a source of unit voltage and all the other ports are short-circuited. We note that $k_{i(k),j}$ is equal to some element of the k-th row of K_{ij}, $j \neq i$.

The following equations have been obtained in [2]:

$$S_{i_k j} = S_{ij} k_{i(k),j} + \sum_{\substack{m=1 \\ m \neq i,j}}^{p} S_{jm}(k_{i(k),j} - k_{i(k),m});\; S_{i_0 j} = S_{ij} - \sum_{k=1}^{n_i} S_{i_k j}$$

$$1 \leqq i \leqq p,\; j \neq i,\; 1 \leqq k \leqq n_i. \tag{2}$$

Let the column matrices of $S_{i_k j}$'s and S_{ij}'s be denoted by $\bar{S}$ and S respectively. Let

$$\bar{S} = PS, \tag{3}$$

where the matrix P can be obtained using Eq. (2). We note that each entry of P is a linear combination of some potential factors $k_{i(k),j}$'s. The column vector S will be considered non-trivial if the network obtained after short-circuiting all the ports of N is connected. The procedure for K-matrix synthesis given in [2] is essentially contained in the following theorem.

Theorem 1

Let each connected part of the port configuration T appropriate to a given potential factor matrix K be a lagrangian tree. The matrix K can be realized by a resistive n-port network containing no negative conductances if and only if there exists a nontrivial S such that $S \geqq 0$ and $PS \geqq 0$.

Once a suitable value for S satisfying the conditions stipulated in Theorem 1 is obtained, the conductances of the padding network N_p having this value for S and the given K-matrix can be determined [2]. A suitable N_d can then be obtained such that the parallel combination N of N_d and N_p contains no negative conductances. The conductances of one such N_d are given as follows [3]:

$$(g_{i_k j_m})_d = -(g_{i_k j_m})_p + \frac{S_{i_k j} S_{j_m i}}{S_{ij}},\quad j \neq i, \tag{4}$$

$$(g_{i_k i_m})_d = -(g_{i_k i_m})_p.$$

Since

$$(g_{i_k j_m}) = (g_{i_k j_m})_d + (g_{i_k j_m})_p \text{ for all } i \text{ and } j \tag{5}$$

we get from (4) and (5) the following

$$g_{i_k j_m} = \frac{S_{i_k j} S_{j_m i}}{S_{ij}}, \; j \neq i; \; g_{i_K i_m} = 0. \tag{6}$$

Thus we observe from the above discussions, that the synthesis of the K-matrix of a resistive n-port network is essentially equivalent to determining a suitable value for S satisfying the conditions stipulated in Theorem 1. Onec S and hence $\bar{S}$ are determined, the conductances of an n-port network realizing the given K-matrix will be given by (6).

2. Synthesis of the K-matrix of $(n+3)$-node resistive n-port networks

In this section we establish a set of necessary and sufficient conditions which the elements of the submatrices K_{ij}'s, $j \neq i$, should satisfy in order that the matrix

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

be realizable as the K-matrix of an $(n+3)$-node resistive n-port network containing no negative conductances. The notation introduced in the last section will be followed. We assume that each T_i is a lagrangian tree.

It may be noted that the potential factors of the required network should satisfy the inequalities

$$0 \leqq k_{i(k),j} \leqq 1 \quad \text{and} \quad \sum_{k=1}^{n_i} k_{i(k),j} \leqq 1. \tag{7}$$

It may be clear from the discussions of the previous section that the conditions we are seeking for the realizability of the matrix K are the same as the conditions which the potential factors $k_{i(k),j}$'s should satisfy so that all the $S_{i_k j}$'s of the required n-port network are non-negative.

Consider the following equations obtained using (2):

$$S_{2_k 1} = S_{12} k_{2(k),1} + S_{13}(k_{2(k),1} - k_{2(k),3}), \quad k = 1, 2, \ldots, n_2,$$
$$S_{2_0 1} = S_{12}\left(1 - \sum_{k=1}^{n_2} k_{2(k),1}\right) - S_{13} \sum_{k=1}^{n_2} (k_{2(k),1} - k_{2(k),3}). \tag{8}$$

The above equations are of the following form:

$$S_{2_k 1} = A_k S_{12} + B_k S_{13}, \quad k = 0, 1, \ldots, n_2. \tag{9}$$

It may be observed that A_k's and B_k's are functions of potential factors and can be evaluated easily. Further we also have, by virtue of (7) that all A_k's are non-negative. We have to choose S_{12} and S_{13} so that $S_{2_k 1}$'s are non-negative. That is

$$S_{2_k 1} = A_k S_{12} + B_k S_{13} \geqq 0. \tag{10}$$

If all B_k's are non-negative then any non-negative values for S_{12} and S_{13} will satisfy (10). If for any k, B_k is negative then we require that

$$A_k S_{12} - |B_k| \, S_{13} \geqq 0; \quad \text{i.e.} \quad \frac{S_{12}}{S_{13}} \geqq \frac{|B_k|}{A_k}. \tag{11}$$

Hence, we conclude that $S_{2_k 1}$'s obtaind using Eq. (8) and choosing values for S_{12} and S_{13} satisfying the following inequality will be non-negative.

$$\frac{S_{12}}{S_{13}} \geqq L_1, \tag{12}$$

where $L_1 = \text{Max}\left\{\frac{|B_k|}{A_k}, B_k < 0\right\}$. It may be noted that if L_1 is not finite then S_{13} should be chosen zero and S_{12} may be given any non-negative value.

Consider next the following equations also obtained using (2)

$$S_{3_k1} = S_{13}k_{3(k),1} + S_{12}(k_{3(k),1} - k_{3(k),2}), \quad k = 1, \dots, n_3,$$

$$S_{3_01} = S_{13}\left(1 - \sum_{k=1}^{n_3} k_{3(k),1}\right) - S_{12}\sum_{k=1}^{n_3}(k_{3(k),1} - k_{3(k),2}). \tag{13}$$

The above equations are of the following form:

$$S_{3_k1} = C_kS_{13} + D_kS_{12}; \quad k = 0, 1, \dots, n_3. \tag{14}$$

We again note that all C_k's are non-negative and C_k's and D_k's are functions of potential factors.

If all D_k's are non-negative, any non-negative values for S_{13} and S_{12} will make all S_{3_k1} non-negative. If for any k, D_k is negative, then we need that

$$C_kS_{13} - |D_k|\, S_{12} \geqq 0,$$

i.e.

$$\frac{S_{12}}{S_{13}} \leqq \frac{C_k}{|D_k|}. \tag{15}$$

Hence we get that

$$\frac{S_{12}}{S_{13}} \leqq U_1, \tag{16}$$

where $U_1 = \text{Min}\left\{\frac{C_k}{|D_k|}, D_k < 0\right\}$. If U_1 is zero then S_{12} should be chosen zero and S_{13} may be given any non-negative value.

Similarly considering S_{1_k2}'s and S_{3_k2}'s as well as S_{1_k3}'s and S_{2_k3}'s in succession we can get the upper and lower limits for $\frac{S_{12}}{S_{23}}$ and $\frac{S_{13}}{S_{23}}$ respectively. These inequalities along with (12) and (16) may be written as follows:

$$\text{(a)}\quad U_1 \geqq \frac{S_{12}}{S_{13}} \geqq L_1; \quad \text{(b)}\quad U_2 \geqq \frac{S_{12}}{S_{23}} \geqq L_2; \quad \text{(c)}\quad U_3 \geqq \frac{S_{13}}{S_{23}} \geqq L_3. \tag{17}$$

We have to choose S_{12}, S_{13} and S_{23} so that the above inequalities are satisfied. Such a selection will be possible only if these inqualities are consistent. It may be noted that consistency of these inequalities requires that all U_k's and L_k's be positive and finite. Further, we need that

$$U_1 \geqq L_1; \quad U_2 \geqq L_2; \quad U_3 \geqq L_3, \tag{18}$$

$$\frac{U_2}{L_3} \geqq \frac{S_{12}}{S_{13}} \geqq \frac{L_2}{U_3}. \tag{19}$$

Let the set of values lying between and including U_1 and L_1 be denoted by R_1 and the set of values lying between and including $\frac{U_2}{L_3}$ and $\frac{L_2}{U_3}$ by R_2. Let $R = R_1 \cap R_2$.

Inequallties (17a) and (19) require that R be non-empty. If inequalities (18) are satisfied and R is non-empty then we can always choose S_{12}, S_{23} and S_{13} so that (17) is satisfied.

Based on these discussions, we give the following procedure for realizing the given K-matrix.

Step 1. Calculate L_i's and U_i's.

Step 2. Let $\frac{S_{12}}{S_{13}} = r$, $r \in R$. Choose an arbitrary positive value for S_{13} and evaluate $S_{12} = rS_{13}$.

Step 3. Choose a positive value for $\frac{S_{12}}{S_{23}}$ such that $U_2 \geqq \frac{S_{12}}{S_{23}} \geqq L_2$ and obtain S_{23}.

Step 4. Using the values of S_{12}, S_{13} and S_{23} obtained in the previous step, calculate $S_{i_k j}$'s using Eq. (2).

Step 5. Obtain the edge conductances of the required n-port network using Eq. (6).

In some cases we may not be able to evaluate L_k or U_k for some k. In such cases, we have to choose S_{12}, S_{13} and S_{23} using the remaining inequalities. The above discussions can be modified suitably to handle such cases.

We now summarize the results of this section in the following theorem, wherein we have assumed that all U_k's and L_k's are determinable, i.e., the inequallties (17) are available.

Theorem 2

Let an $(n \times n)$-matrix $K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$ of real non-negative elements be given. Let K_{ii}'s contain only 0's and 1's. Let the port-configuration T_i, $i = 1, 2, 3$, defined by K_{ii} be a lagrangian tree. The matrix K is realizable as the potential factor matrix of an $(n+3)$-node resistive n-port containing no negative conductances if and only if

i) a) all elements in any row of K_{ij}, $j \neq i$ are equal

b) $0 \leqq k_{i(k),j} \leqq 1$, $i, j = 1, 2, 3$; $k = 1, \ldots, n_i$

c) $\sum_{k=1}^{n_i} k_{i(k),j} \leqq 1$, $i, j = 1, 2, 3$, $j \neq i$

ii) a) $U_1 \geqq L_1$, $U_2 \geqq L_2$, $U_3 \geqq L_3$

b) R is non-empty.

It may be mentioned that the type of port-configuration used in the above theorem does not result in any loss of generality.

We now illustrate the results by an example.

Example

Let it be required to realize the matrix

$$K = \begin{bmatrix} 1 & 0.8 & 0.7 \\ 0.6 & 1 & 0.4 \\ 0.3 & 0.4 & 1 \end{bmatrix}$$

as the potential factor matrix of an 3-port network containing no negative conductances.

The port configuration T corresponding to the matrix K is obtained as shown in Fig. 2.

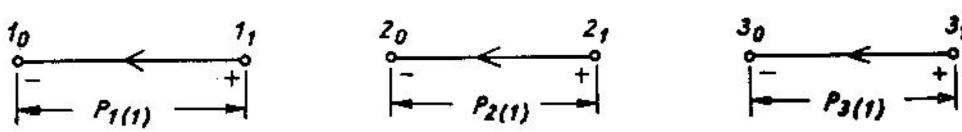

Fig. 2. The port configuration T corresponding to the matrix K

The potential factors $k_{i(k),j}$'s are identified as follows:

$$k_{1(1),2} = 0.8, \quad k_{1(1),3} = 0.7, \qquad k_{2(1),1} = 0.6,$$
$$k_{2(1),3} = 0.4, \quad k_{3(1),1} = 0.3 \quad \text{and} \quad k_{3(1),2} = 0.4.$$

Using (2) the following equations relating S_{i_kj}'s and S_{ij}'s are obtained

$$S_{1_02} = 0.2S_{12} - 0.1S_{23},$$
$$S_{1_12} = 0.8S_{12} + 0.1S_{23},$$
$$S_{1_03} = 0.3S_{13} + 0.1S_{23},$$
$$S_{1_13} = 0.7S_{13} - 0.1S_{23},$$
$$S_{2_01} = 0.4S_{12} - 0.2S_{13},$$
$$S_{2_11} = 0.6S_{12} + 0.2S_{13},$$
$$S_{2_03} = 0.6S_{23} + 0.2S_{13},$$
$$S_{2_13} = 0.4S_{23} - 0.2S_{13},$$
$$S_{3_01} = 0.7S_{13} + 0.1S_{12},$$
$$S_{3_11} = 0.3S_{13} - 0.1S_{12},$$
$$S_{3_02} = 0.6S_{23} - 0.1S_{12},$$
$$S_{3_12} = 0.4S_{23} + 0.1S_{12}.$$

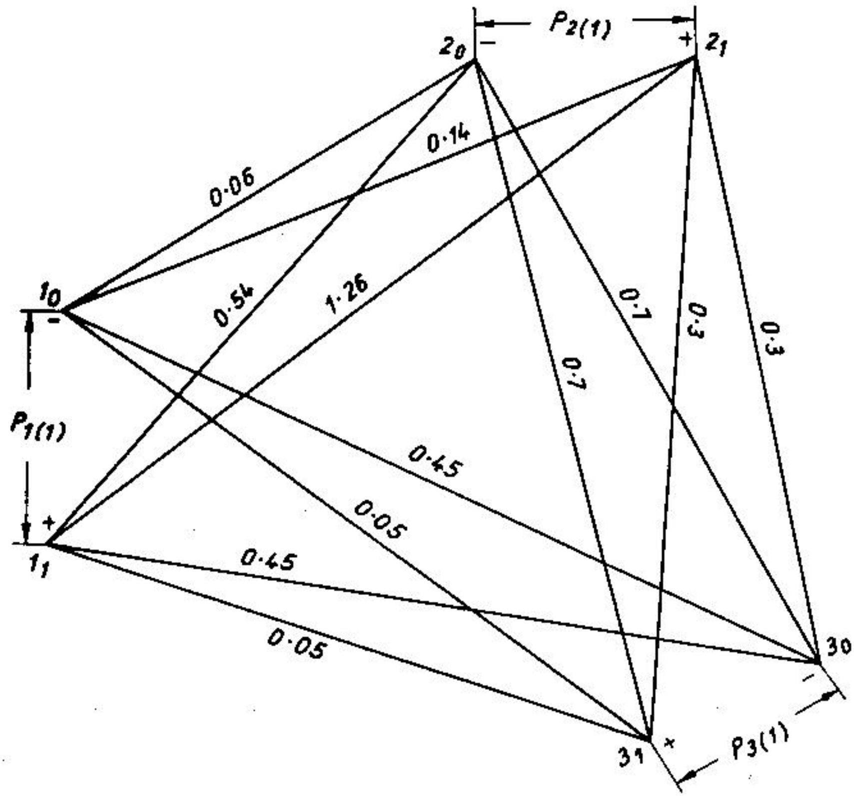

Fig. 3. The network N of the example

From the above the following constraints are obtained

$$L_1 = 0.5 \leqq \frac{S_{12}}{S_{13}} \leqq 3 = U_1,$$

$$L_2 = 0.5 \leqq \frac{S_{12}}{S_{23}} \leqq 6 = U_2,$$

$$L_3 = 1/7 \leqq \frac{S_{13}}{S_{23}} \leqq 2 = U_3.$$

It may be noted that $\frac{U_2}{L_3} = 42$ and $\frac{L_2}{U_3} = 0.25$. Hence R is non-empty and contains all the values lying between and including 0.5 and 3. Choosing $S_{13} = 1$ and $\frac{S_{12}}{S_{13}} = 2$ and $\frac{S_{12}}{S_{23}} = 1$ we get $S_{12} = 2$ and $S_{23} = 2$. All the $S_{i_k j}$'s are then obtained using these values of S_{ij}'s in Eq. (2). The conductances of an $(n+3)$-node n-port network N realizing the given K-matrix are then obtained using (6). The network N is shown in Fig. 3.

3. Conclusions

The simple set of necessary and sufficient conditions derived in this paper for the realization of the K-matrices of $(n+3)$-node resistive n-port networks underlines the usefulness of the procedure in [2] for the realization of the K-matrices of $(n+p)$ node n-port networks. The results of this paper find application in the realization of the sensitivity coefficient matrix [4].

Abstract

A simple set of necessary and sufficient conditions is established for the synthesis of the K-matrix of an $(n+3)$-node resistive n-port network.

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