SENSITIVITY INVARIANTS FOR ACTIVE LUMPED/DISTRIBUTED NETWORKS

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The invariant nature of the sum of the sensitivities of network functions of a general linear active lumped/distributed network is established. Results of Holt and Fidler on summed sensitivities are extended to networks containing tapered lines, v.c.t.s and c.v.t.s simultaneously.

In this letter, we establish the invariant nature of the sum of the sensitivities of the network functions of a general active lumped/distributed network. We also extend the results of Holt and Fidler on summed sensitivity invariants. We consider a network N, consisting of lumped capacitors ($C_l = 1/D_l$), inductors ($L_l = 1/\Gamma_l$), resistors ($R_l = 1/G_l$), gyrators, controlled sources, negative-impedance convertors and RC and LC tapered lines. Let N^A denote the adjoint of N. Then we have

$$\sum V_e \phi_e = \sum_p \phi_p V_p \quad \quad (1)$$

where $V_e(\phi_e)$ and $V_p(\phi_p)$ are, respectively, the voltage across (currents through) interior elements (or ports of interior elements) and external ports for N and N^A. If Z_{ij} is ijth element of the impedance matrix Z of N, from the results of Reference 1, it can be shown that

$$\frac{\partial Z_{ij}}{\partial p_k} = \phi_e{}^t \frac{\partial Z_e}{\partial p_k} I_e \quad \quad (2)$$

and

$$\frac{\partial Z_{ij}}{\partial p_k} = -\psi_e{}^t \frac{\partial Y_e}{\partial p_k} V_e \quad \quad (3)$$

where $\phi_e(I_e)$ and $\psi_e(V_e)$ are the column vectors of currents through and voltages across the element port(s) in $N^A(N)$, respectively, and are evaluated with the ith (jth) port of $N^A(N)$ excited with unit current, all other ports being open-circuited.

If p_k is identified with the value of a resistor, an inductor, inverse capacitance or transfer resistance of a c.v.t. ($r_l = 1/g_l$), it is seen that

$$p_k \frac{\partial Z_{ij}}{\partial p_k} = \phi_e{}^t V_e \quad \quad (4)$$

For a v.c.t., Z_e does not exist but Y_e does. It is therefore easily seen that

$$\delta_l \frac{\partial Y_e}{\partial \delta_l} = Y_e \quad \quad (5)$$

where $\delta_l = 1/\gamma_l$ is the transfer conductance of the v.c.t. Hence, from eqn. 3,

$$\delta_l \frac{\partial Z_{ij}}{\partial \delta_l} = -\phi_e{}^t V_e$$

or

$$\gamma_l \frac{\partial Z_{ij}}{\partial \gamma_l} = \phi_e{}^t V_e \quad \quad (6)$$

For the gyrator, $Z_{11} = Z_{22} = 0$, $Z_{12} = \alpha_{l1}$ and $Z_{21} = \alpha_{l2}$, and it can be shown, using eqn. 2, that

$$\alpha_{l1} \frac{\partial Z_{ij}}{\partial \alpha_{l1}} + \alpha_{l2} \frac{\partial Z_{ij}}{\partial \alpha_{l2}} = \phi_e{}^t V_e \quad \quad (7)$$

For an RC tapered line with resistance $R_l = R_{0l} f(x)$ and capacitance $C_l = C_{0l} g(x)$, it is known that[2]

$$Z_e = R_{0l} F \quad \quad (8)$$

where each element of the 2×2 matrix F is a function of the product

$$\left(sR_{0l} C_{0l} = sR_{0l} \frac{1}{D_{0l}}\right)$$

It can be easily shown that

$$R_{0l} \frac{\partial Z_e}{\partial R_{0l}} + D_{0l} \frac{\partial Z_{ij}}{\partial D_{0l}} = \phi_e{}^t V_e \quad \quad (9)$$

It can also be shown that

$$Z_{RCl} \frac{\partial Z_{ij}}{\partial Z_{RCl}} = \phi_e{}^t V_e \quad \quad (10)$$

where $Z_{RCl} = \sqrt{(R_{0l}/C_{0l})}$ is the characteristic impedance of a tapered RC line.

Similarly, for an LC tapered line with inductance $L_l = L_{0l} f(x)$,

$$L_{0l} \frac{\partial Z_{ij}}{\partial L_{0l}} + D_{0l} \frac{\partial Z_{ij}}{\partial D_{0l}} = \phi_e{}^t V_e \quad \quad (11)$$

and

$$Z_{LCl} \frac{\partial Z_{ij}}{\partial Z_{LCl}} = \phi_e{}^t V_e \quad \quad (12)$$

where $Z_{LCl} = \sqrt{(L_{0l}/C_{0l})}$ is the characteristic impedance of a lossless tapered line.

It can be shown that, for v.v.t.s, c.c.t.s, v.n.i.c.s and c.n.i.c.s,

$$\phi_e{}^t V_e = 0 \quad \quad (13)$$

If p_k is a member of the set

$$\{p_k\} = \{R_l, L_l, D_l(\alpha_{l1}, \alpha_{l2}), (R_{0l}\, D_{0l}), Z_{LCl}, r_l, \gamma_l\}$$

where the subset (R_{0l}, D_{0l}) and Z_{LCl} may be replaced by Z_{RCl} and (L_{0l}, D_{0l}), respectively, then for N_1, using eqn. 1, we obtain

$$\sum_{p_k} \frac{p_k}{Z_{ij}} \frac{\partial Z_{ij}}{\partial p_k} = 1 \quad \quad (14)$$

i.e. the sum of sensitivities of Z_{ij} is a constant independent of Z_{ij}. Similarly, it can be shown that

$$\sum_{p_k} \frac{p_k}{Y_{ij}} \frac{\partial Y_{ij}}{\partial p_k} = -1 \quad \quad (15)$$

$$\sum_{p_k} \frac{p_k}{T} \frac{\partial T}{\partial p_k} = 0 \quad \quad (16)$$

and

$$\sum_{p_k} p_k \frac{\partial S}{\partial p_k} = \tfrac{1}{2}(U - S^2) \quad \quad (17)$$

where T is a voltage or current transfer function, and U and S are unit and scattering matrices, respectively. Eqn. 17 is obtained using the procedure followed in Reference 3. If normalising resistors $\{\rho_l\}$ are also included in $\{p_k\}$ to obtain the set $P' = \{p_k'\}$, it can be shown that the sum of the sensitivities of any scattering parameter or the group delay over the set P' is zero.

Since it is more usual to calculate sensitivity with respect to C_l rather than D_l, Holt and Fidler[4] have defined summed sensitivity for a class of active lumped networks. We now extend these results to networks containing v.c.t.s and c.v.t.s as well as tapered RC and LC lines. Consider a network N, consisting of lumped resistors, capacitors, gyrators, RC tapered lines, controlled sources and n.i.c.s. Let the set $Q = \{q_k\} = \{q_{k1}\} \cup \{q_{k2}\}$, $\{q_{k1}\} = \{R_l, (\alpha_{l1}, \alpha_{l2}), R_{0l}, r_l, \gamma_l\}$ and $\{q_{k2}\} = \{C_l, C_{0l}\}$. Then the summed sensitivity of Z_{ij} over the set $\{q_k\}$ is defined as

$$\sum_{q_k} \frac{q_k}{Z_{ij}} \frac{\partial Z_{ij}}{\partial q_k} = \sum_{q_{k1}} \frac{q_{k1}}{Z_{ij}} \frac{\partial Z_{ij}}{\partial q_{k1}} + \sum_{q_{k2}} \frac{q_{k2}}{Z_{ij}} \frac{\partial Z_{ij}}{\partial q_{k2}} \quad \quad (18)$$

It should be noted that the sensitivity sum considered in eqn. 14 is

$$\sum_{q_{k1}} \frac{q_{k1}}{Z_{ij}} \frac{\partial Z_{ij}}{\partial q_{k1}} - \sum_{q_{k2}} \frac{q_{k2}}{Z_{ij}} \frac{\partial Z_{ij}}{\partial q_{k2}} \quad \quad (19)$$

If $\dot{q}_k$ is identified with a lumped capacitor C_l,

$$q_{k2}\frac{\partial Z_{ij}}{\partial q_{k2}} = -\phi_e^t Z_e I_e \quad \text{if} \quad q_{k2} \in Z_e \quad . \quad . \quad . \quad . \quad . \quad (20)$$

For an RC tapered line, using eqn. 8,

$$C_{0l}\frac{\partial Z_e}{\partial C_{0l}} = sR_{0l}^2 C_{0l}\dot{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where the dot denotes differentiation with respect to $sR_{0l}C_{0l}$. Thus, from eqns. 20 and 21,

$$\sum_{q_{k2}} q_{k2}\frac{\partial Z_{ij}}{\partial q_{k2}} = -\sum_{\mathrm{I}} \phi_e^t Z_e I_e + \sum_{\mathrm{II}} \phi_e^t sR_{0l}^2 C_{0l}\dot{F}I_e \quad (22)$$

where the sum $\sum_{\mathrm{I}}\left(\sum_{\mathrm{II}}\right)$ consists of all terms due to lumped capacitors (capacitances of RC tapered lines).

If a complex frequency variable s is treated as a parameter, it follows from the results of Reference 1 that

$$s\frac{\partial Z_{ij}}{\partial s} = \sum_e \phi_e^t s\frac{\partial Z_e}{\partial s} I_e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The contribution to the right-hand side of eqn. 23 due to all resistors, gyrators, c.v.t.s and v.c.t.s is zero. The contribution from lumped capacitors is

$$-\sum_{\mathrm{I}} \phi_e^t Z_e I_e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

while, for RC lines, from eqn. 8,

$$s\frac{\partial Z_e}{\partial s} = sR_{0l}^2 C_{0l}\dot{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Thus, from eqns. 22, 23, 24 and 25, we obtain

$$\sum_{q_{k2}} \frac{q_{k2}}{Z_{ij}}\frac{\partial Z_{ij}}{\partial q_{k2}} = \frac{s}{Z_{ij}}\frac{\partial Z_{ij}}{\partial s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Thus, from eqns. 14, 19 and 26,

$$\sum_{q_{k1}} \frac{q_{k1}}{Z_{ij}}\frac{\partial Z_{ij}}{\partial q_{k1}} + \sum_{q_{k2}} \frac{q_{k2}}{Z_{ij}}\frac{\partial Z_{ij}}{\partial q_{k2}} = \frac{2s}{Z_{ij}}\frac{\partial Z_{ij}}{\partial s} + 1 \quad . \quad . \quad (27)$$

Hence the summed sensitivity of Z_{ij} over $\{q_k\}$ is an invariant and is independent of the realisation of Z_{ij}.

Similarly, it can be shown that

$$\sum_{q_k} \frac{q_k}{Y_{ij}}\frac{\partial Y_{ij}}{\partial q_k} = \frac{2s}{Y_{ij}}\frac{\partial Y_{ij}}{\partial s} - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$\sum_{q_k} \frac{q_k}{T}\frac{\partial T}{\partial q_k} = \frac{2s}{T}\frac{\partial T}{\partial s} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Next, consider a network N_2 consisting of lumped capacitors, inductors, lossless tapered lines, v.v.t.s, c.c.t.s and n.i.c.s. Let the set $M = \{m_k\}$ be $\{L_l, C_l, (L_{0l}, C_{0l})\}$. Then the summed sensitivity of a network function over the set M is given by

$$\sum_{m_k} \frac{m_k}{F}\frac{\partial F}{\partial m_k} = \frac{s}{F}\frac{\partial F}{\partial s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where F can be an impedance parameter, an admittance parameter or a voltage or current transfer function.

The basis-free normalised scattering matrix S for a linear time-invariant network with a resistive reference is given by[3]

$$S = U - 2\rho^{\frac{1}{2}}(Z+\rho)^{-1}\rho^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

For network N_1, the summed sensitivity of S over the set $\{q_k\}$ could be obtained by using eqns. 14 and 31 as

$$\sum_{q_k} q_k\frac{\partial S}{\partial q_k} = 2s\frac{\partial S}{\partial s} + \tfrac{1}{2}(U - S^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

If the set $\{\rho_l\}$ of normalising resistors is also included,

$$\sum_{q_{k1}} q_{k1}\frac{\partial S}{\partial q_{k1}} - \sum_{q_{k2}} q_{k2}\frac{\partial S}{\partial q_{k2}} + \sum_{\rho_l} \rho_l\frac{\partial S}{\partial \rho_l} = 0 \quad . \quad . \quad (33)$$

It could be shown, using eqns. 17, 32 and 33, that

$$\sum_{q_k'} q_k'\frac{\partial S}{\partial q_k'} = 2s\frac{\partial S}{\partial s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where

$$\{q_k'\} = \{q_{k1}\} \cup \{q_{k2}\} \cup \{\rho_l\} = \{q_k\} \cup \{\rho_l\}$$

Thus the summed sensitivity of the scattering matrix over the set $Q' = \{q_k'\}$ is independent of the realisation.

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