Unimodular Property of Fundamental Cut-Set and Circuit Matrices

The unimodular property of the fundamental cut-set and circuit matrices Q_f and B_f of a linear graph has been established earlier in Refs. 1, 2, 3. In this correspondence we give a new graph-theoretic proof of this property.

Consider a connected graph G and the fundamental cut-set matrix Q_f of G with respect to a tree T. Let the branches of T be denoted by $b_1, b_2, \ldots, b_{v-1}$ where v is the number of vertices of G. Let row i of Q_f correspond to the cut-set q_i with respect to the branch b_i of T.

Consider the graph G_i^o obtained from G after coalescing the vertices of b_i and removing all the self loops formed due to the coalescing. G_i^o will contain $v-1$ vertices. Let the set of edges of G which form self loops when the vertices of b_i are coalesced be denoted by S. Then it is obvious that $b_i \in S$. Also, $b_j \notin S$, $j \neq i$, since b_j and b_i are not in parallel in G. Let the subgraph of G_i^o consisting of the edges of $(T - b_i)$ be denoted by T_i^o. T_i^o consists of $v-2$ edges. Further it contains no circuits since coalescing the vertices of any branch of a tree does not produce any circuit consisting of the remaining branches only. Hence T_i^o is a tree of G_i^o. (Theorem 2.10 of Ref. 1.)

Consider any cut-set q_j, $j \neq i$ of G. Let the removal of the edges of q_j from G partition the vertices of G into two sets A_i and B_i. Since $b_i \notin q_j$ for $j \neq i$, both the vertices of b_i will be either in A_i or in B_i. Hence the edges of q_j do not form self loops, when the vertices of b_i are coalesced. Hence $q_j \subseteq G_i^o$. Also q_j is a cut-set of G_i^o since coalescing the vertices of any connected graph does not affect its connectivity. Thus the cut-sets $q_1, q_2, \ldots, q_{i-1}, q_{i+1}, \ldots, q_{v-1}$ of G are also cut-sets for G_i^o. Each one of these cut-sets contains one and only one branch of T_i^o and they together include all the branches of T_i^o. Hence these cut-sets form the set of fundamental cut-sets of G_i^o with respect to T_i^o. Let the corresponding cut-set matrix be Q_i^o.

Let the subgraph Q_{f-i} obtained from Q_f after removing row i, be partitioned as $Q_{f-i} = [Q^o_{f-i} | X]$ where the columns of Q^o_{f-i} correspond to the edges in G^o_i. Since the cut-sets of G corresponding to the rows of Q_{f-i} are the same as those of G^o_i corresponding to the rows of Q^o_i, we can conclude that

$$Q^o_{f-i} = Q^o_i \quad (1)$$

Further, no column in Q^o_{f-i} is zero since Q^o_i covers the graph G^o_i. The edge corresponding to any column in X is not present in any of the cut-sets $q_1, q_2, \ldots, q_{i-1}, q_{i+1}, \ldots q_{v-1}$. Hence each column in X is zero. Thus $X = 0$. Hence the zero columns in Q_{f-i} are in one-to-one correspondence with the edges which form self loops when the vertices of b_i are coalesced. Hence in view of equation (1) the matrix Q^o_{f-i} obtained from Q_f after (i) removing row i, and (ii) removing resultant zero columns, is a fundamental cut-set matrix of a connected graph.

The above arguments can be continued to show that the matrix $Q^o_{f-i-j-k\ldots}$ obtained after removing rows $i, j, k, \ldots$ of Q_f and also removing the resultant zero columns is a fundamental cut-set matrix of a connected graph.

Theorem: Q_f and B_f are unimodular.

Proof: Let Q_f be the fundamental cut-set matrix of a graph G with respect to a tree T and let A be a reduced incidence matrix of G. Let A and Q_f be partitioned as

$$Q_f = [u \mid Q_{fc}] \quad (2)$$

and

$$A = [A_T \mid A_C] \quad (3)$$

where the columns of u and A_T correspond to the branches of T.

Since any row in a cut-set matrix can be expressed as a linear combination of the rows of A, Q_f can be written as

$$Q_f = DA \quad (4)$$

It then follows from (2), (3) and (4) that

$$D = A_T^{-1} \quad (5)$$

Since A is unimodular[1], A_T and hence $A_T^{-1} = D$ are unimodular. Consider any non-singular submatrix Q_{11} of Q_f and of order $v-1$. Let the corresponding submatrix of A be denoted by A_{11}. Then we have

$$Q_{11} = DA_{11}$$

Since D and A_{11} are both unimodular

$$\det[Q_{11}] = +1 \text{ or } -1$$

Thus any $(v-1)\times(v-1)$ non-singular submatrix of Q_f has a determinant equal to 1 or −1.

Consider any non-singular $(k \times k)$ submatrix Q^k of Q_f. Let Q^k be formed from the first k rows of Q_f. Since Q^k is non-singular, no column of Q^k is zero. Hence Q^k is a submatrix of $Q^o_{f-(k+1)-(k-2)\ldots(v-1)}$. But $Q^o_{f-(k+1)-(k+2)\ldots(v-1)}$ is a fundamental cut-set matrix of a connected graph with $(k+1)$ vertices. Hence in view of the earlier arguments, Q^k has a determinant equal to 1 or -1.

Thus every non-singular submatrix of Q_f has a determinant equal to 1 or -1. Hence Q_f is unimodular.

Since Q_f is unimodular, Q_{fc} and hence Q_{fc}^T are also unimodular. But B_f, the fundamental circuit matrix of G with respect to T is given by

$$B_f = [-Q_{fc}^t \mid u]$$

Since any matrix $[E \mid u]$ is unimodular if E is unimodular, we conclude that B_f is also unimodular. Thus the fundamental cut-set and circuit matrices of a graph are unimodular.

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References

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