A theorem in the theory of determinants and the number of spanning trees in a graph

L'utilisation d'un théorème de la théorie des déterminants pour établir le nombre d'embranchements dans un système

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A network-theoretic approach for counting the number of spanning trees of a graph is proposed. This approach is based on a theorem in the theory of determinants. Following this approach, a recurrence relation for counting Γ_n, the number of spanning trees in a multigraph ladder having $(n + 1)$ nodes, is established. We then obtain a recurrence relation connecting the sequences $\{W_n\}$ and $\{\Gamma_n\}$ where W_n is the number of spanning trees in a multigraph wheel having $(n + 1)$ nodes. The significance of the approach is further illustrated by giving simple proofs of certain well-known results, in particular, the formula for counting the number of spanning trees in a cascade of 2-port networks.

La présente communication a pour but de soumettre une démarche théorique et systémique pour compter le nombre d'embranchements dans un graphique. Cette démarche est fondée sur un théorème faisant partie de la théorie des déterminants. Il s'agit d'abord d'établir une relation de fréquence en désignant par Γ_n le nombre d'embranchements dans un système comportant (n + 1) noeuds. Nous obtenons ainsi une relation de fréquence entre les séquences $[W_n]$ et $[\Gamma_n]$ où W_n constitue le nombre d'embranchements dans un système comportant $(n + 1)$ noeuds. La pertinence de cette démarche est ensuite établie au moyen de preuves simples utilisant certains résultats bien connus, comme, en particulier, la formule pour établir le nombre d'embranchements dans une cascade de systèmes comportant deux sources.

Introduction

Enumeration of the spanning trees of a graph has been a problem of considerable interest to network theorists. In 1967, Myers established a recurrence relation for enumerating spanning trees in a cascade of 2-port networks.[1] Recurrence relations for enumerating spanning trees in wheels and multigraph wheels have been established by Myers[2] and Bose, Feick and Sun.[3] Myers has also related the spanning tree enumeration problem to that of enumerating partitions of an integer.[4,5] In addition, Myers has presented, in reference 4, two enumerating functions for counting spanning trees. One of these functions is derived by removing an incidence set from the given graph, and the other is derived by removing two disjoint incidence sets of a certain type.

In this paper, we propose a network-theoretic approach for enumerating spanning trees. Our approach is based on a theorem in the theory of determinants. In a later section, we obtain a recurrence relation for the number of spanning trees in a multigraph ladder. We also relate the sequences $\{W_n\}$ and $\{\Gamma_n\}$ where W_n and Γ_n are the numbers of spanning trees in a multigraph wheel and a ladder, each having $(n + 1)$ nodes. We conclude by giving a simple proof of the results of reference 1. This proof is also based on the approach given in the next section. Unless otherwise stated, we follow the notation given in reference 6.

A network-theoretic approach for enumerating spanning trees

It is well known that the number of spanning trees in a connected $(n + 1)$-vertex graph G is equal to the determinant of the matrix AA^t where A is any n-rowed submatrix of the incidence matrix of G.[6] Suppose N is an electrical network having a graph which is identical to G. Further suppose that each network element of N is a conductance of 1 siemen. Then the determinant of the matrix AA^t is also equal to the sum of the tree-admittance-products of N. Thus, without any loss of generality, we may assume that our problem is to evaluate the sum of tree-admittance-products of a network each element of which is a conductance of 1 siemen. In this section, we propose a new approach to evaluate this number. Our approach is based on the following theorem.[7]

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Theorem 1:

Let Y be an $(n \times n)$ matrix partitioned as follows:

$$Y = \begin{array}{cc} \begin{array}{cc} \leftarrow n_1 \rightarrow & \leftarrow n_2 \rightarrow \end{array} & \\ \left[\begin{array}{c:c} Y_{11} & Y_{12} \\ \hdashline Y_{21} & Y_{22} \end{array}\right] & \begin{array}{c} \updownarrow n_1 \\ \updownarrow n_2 \end{array} \end{array}$$

Then

$$\det(Y) = \det(Y_{11}).\det(Y_{22} - Y_{21} Y_{11}^{-1} Y_{12})$$

$$= \det(Y_{22}).\det(Y_{11} - Y_{12} Y_{22}^{-1} Y_{21}). //$$

Let the nodes of the network N be numbered as $1,2,3,\ldots,n$, $n + 1$. Let A be the n-rowed submatrix obtained after removing from the incidence matrix of N the row corresponding to node $(n + 1)$. Let the matrix $Y = AA^t$ be partitioned as in Theorem 1 so that the rows and columns of Y_{11} correspond to the n_1 nodes $1,2,\ldots,n_1$ of N, and those of Y_{22} correspond to the n_2 nodess $n_1 + 1$, $n_1 + 2$, $\ldots,n$ of N. Note that the matrix Y is the node-admittance matrix of N with node $(n + 1)$ as datum.

Let K be a subset of the vertex set $V = \{1,2,\ldots,n, n + 1\}$ of N. Then, we shall define the networks N_K and N_K^0 as follows:

N_K = the network that results after short-circuiting all the nodes of N which do not belong to K.

N_K^0 = the network that results after open-circuiting or suppressing all the nodes of N which do not belong to K. (Note: Suppression of a node can be achieved by the generalized star-delta transformation at the node.)

With these definitions, we have the following interpretations for the matrices Y_{11} and $Y_{22} - Y_{22} Y_{11}^{-1} Y_{12}$:

$_{11}$ =. the node-admittance matrix of the network N_K with node $(n + 1)$ as datum,

and

$Y_{22} - Y_{21} Y_{11}^{-1} Y_{12}$ = the node-admittance matrix of the network $K_{\bar{K}}^{0}$ with the node $(n + 1)$ as datum

where $K = \{1,2,\ldots,n\}$ and $\bar{K}$ is the complement of K in V, that is, $\bar{K} = \{n_1 + 1, n_1 + 2,\ldots, n + 1\}$.

From Theorem 1, we obtain the following:

$$T(N) = T(N_K) \bullet T(N_{\bar{K}}^{0}) \qquad (1)$$

where $T(N)$, $T(N_K)$ and $T(N_{\bar{K}}^{0})$ denote respectively the sums of tree-admittance-products of N, N_K and $N_{\bar{K}}^{0}$. Suppose $K = \{1\}$, then $T(N_K)$ is the sum of the conductances of the elements incident on node 1 in N.

Letting

$$T(N_K) = a_1 \qquad (2)$$

we obtain from Eq. (1)

$$T(N) = a_1 \bullet T(N_L^0) \qquad (3)$$

where $L = \{2,3,\ldots,n, n + 1\}$. Note here that $L = V - \{1\}$. Following the same arguments as above we get

$$T(N_L^0) = a_2 \bullet T(N_M^0) \qquad (4)$$

where a_2 is the sum of the conductances incident on node 2 in N_L^0 and $M = \{3,4,5,\ldots,n, n + 1\} = L - \{2\}$. Note that N_M^0 is obtained from N_L^0 by suppressing node 2. This is the same as the network obtained by suppressing nodes 1 and 2 of N.

Eqs. (3) and (4) yield the following:

$$T(N) = a_1 \bullet a_2 \bullet T(N_M^0) \qquad (5)$$

Continuing in this manner, we eventually get

$$T(N) = a_1 \bullet a_2 \bullet \bullet \bullet a_n \qquad (6)$$

where a_1 is as defined in Eq. (2) and for $1 < i \leq n$,

a_i = sum of the conductances of the elements incident on node i in the network $N^0_{\{i,\, i+1,\ldots,\, n+1\}}$.

Note that the application of Eq. (6) requires the determination of a sequence of networks obtained by successive suppression of nodes of N.

Suppose Y_s is the short-circuit admittance matrix of an RLC n-port network N having P as its port-configuration. Let P_o be a spanning tree of N with P as its subgraph. Let N^* be the network that results from N after short-circuiting all the branches of P_o that belong to P. Then we have shown that

$$\det(Y_s) = \frac{T(N)}{T(N^*)}. \qquad (7)$$

In the case of a one-port network, Y_s is the same as the driving point admittance y_{in} and it is easy to see that Eq. (7) generalizes the well-known result for y_{in} in terms of tree-admittance and 2-tree-admittance products.[6]

We wish to add that Eq. (7–27) in reference 8 can be established starting from Theorem 1. We also note that Eq. (6) has proved very useful in a recent work on the computational complexity analysis of a spanning tree enumeration algorithm.[9] In the following sections, we illustrate the application of Eqs. (1) and (6) in deriving certain new results as well as simpler proofs of some known results.

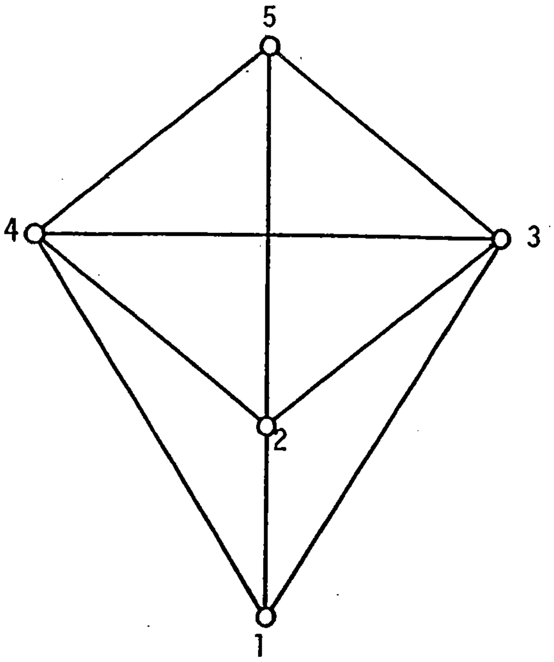

Figure 1: *Graph G.*

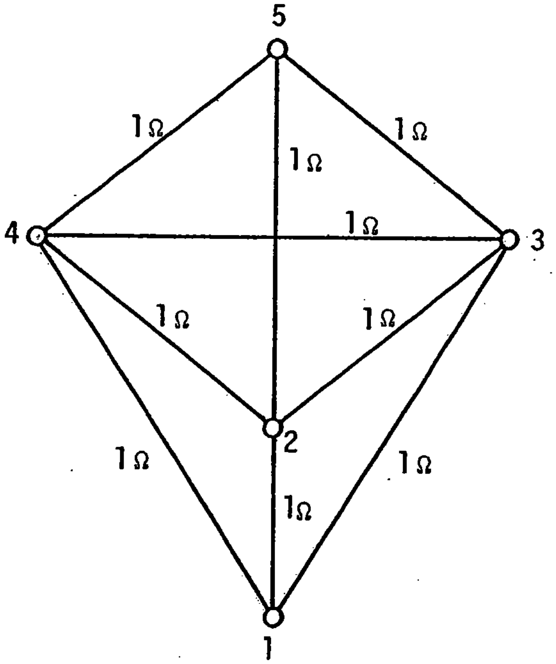

Figure 2: *Network N_1.*

We conclude this section with an example to illustrate the application of Eq. (6). Consider the graph G shown in Figure 1. The number of spanning trees of this graph is the sum of the tree-admittance-products of the resistance network N_1 shown in Figure 2. We shall first compute a_1, a_2, a_3 and a_4 as defined in Eq. (6).

To start with
a_1 = sum of the conductances of resistors connected to node 1 in the network N_1
= 3.

Suppressing node 1 (in other words, performing star-delta transformation at node 1), we get the network N_2 in Figure 3.

a_2 = sum of the conductances of resistors connected to node 2 in N_2
= 11/3.

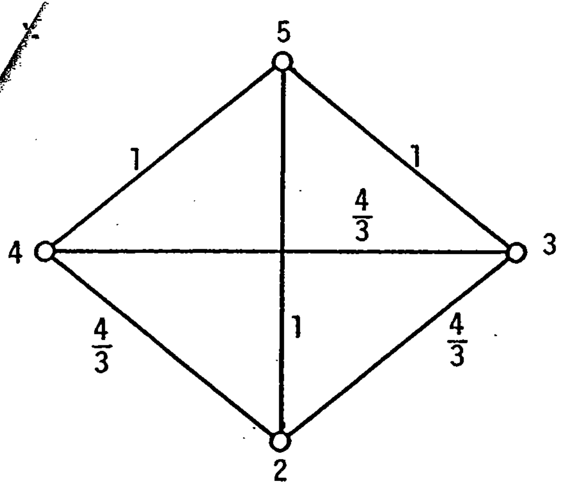

Figure 3: *Network N_2.*

Continuing as above, the networks N_3 and N_4 are obtained and shown in Figure 4 and Figure 5. Also from these networks we get

$$a_3 = \frac{105}{33}$$

and

$$a_4 = \frac{165}{77}.$$

Now using Eq. (6) we get:

Number of spanning trees of G
$= T(N_1)$, sum of tree-admittance-products of N_1
$= a_1 \cdot a_2 \cdot a_3 \cdot a_4$
$= 75$

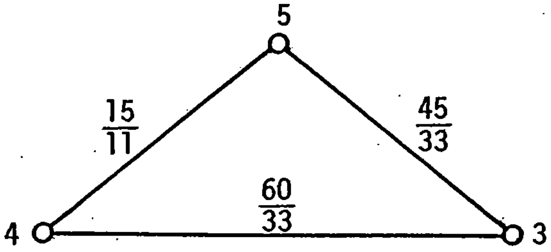

Figure 4: *Network N_3.*

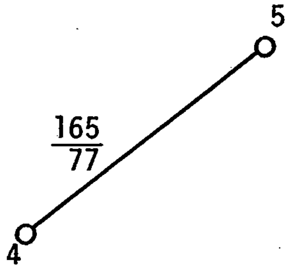

Figure 5: *Network N_4.*

Enumeration of spanning trees in multigraph ladders

Bose, Feick and Sun have obtained in reference 3 a recurrence relation for enumerating the spanning trees of multigraph wheels. They have also obtained an expression for W_n, the number of spanning trees in a multigraph wheel having $(n + 1)$ nodes. We shall establish in this section a recurrence relation for counting the number of spanning trees in multigraph ladders. Thus, our problem is to find an expression for $T(N)$ for the network N shown in Figure 6. In this figure, P and r denote respectively the conductances of the series and shunt arms of the ladder.

We shall follow the same notation that was used in the previous section. Further, for $2 \leq i \leq n$ let,

$$A_i = a_1 \cdot a_2 \cdots a_i \tag{8}$$

and Γ_i = the number of spanning trees in a multigraph ladder having (i + 1) nodes. Note that $\Gamma_n = A_n$

Consider the networks $N^0{}_{\{i,\, i+1, \ldots, n+1\}}$ and $N^0{}_{\{i+1, \ldots, n+1\}}$ shown in Figures 7 and 8. From these, we can obtain the following:

$$Z_{i+1} = \frac{Z_i(P + r) + Pr}{Z_i + P}, 2 \leq i \leq n - 1 \tag{9}$$

$$A_i = A_{i-1}(Z_i + P), \qquad 2 \leq i \leq n \tag{10}$$

and

$$\Gamma_i = A_{i-1} Z_i, \qquad 2 \leq i \leq n \tag{11}$$

Using Eqs. (9)-(11), we obtain

$$\begin{aligned} \Gamma &= A_{n-1} \cdot Z_n \\ &= A_{n-2}(Z_{n-1} + P) \frac{[Z_{n-1}(P + r) + Pr]}{Z_{n-1} + P} \\ &= A_{n-2}\, Z_{n-1}.(P + r) + A_{n-2}\, Pr \\ &= \Gamma_{n-1}(P + r) + A_{n-2} Pr. \end{aligned} \tag{12}$$

Similarly

$$\Gamma_{n-1} = \Gamma_{n-2}(P + r) + A_{n-3} Pr. \tag{13}$$

A recurrence relation connecting Γ_n, Γ_{n-1}, and Γ_{n-2} can be obtained from Eqs. (12) and (13) if we have a relation connecting A_{n-2} and A_{n-3} without involving any Z_i's. Such a relation can be obtained using Eqs. (9) and (10) as follows:

$$\begin{aligned} A_{n-2} &= A_{n-3}(Z_{n-2} + P) \\ &= \Gamma_{n-2} + A_{n-3} P. \end{aligned} \tag{14}$$

Then we obtain from Eqs. (12), (13) and (14) the following recurrence relation to count Γ_n.

$$\Gamma_n = (2P + r)\Gamma_{n-1} - P^2\Gamma_{n-2} \tag{15}$$

The natural choice for Γ_1 and Γ_2 are:

$$\Gamma_1 = r;\ \Gamma_2 = 2Pr + r^2. \tag{16}$$

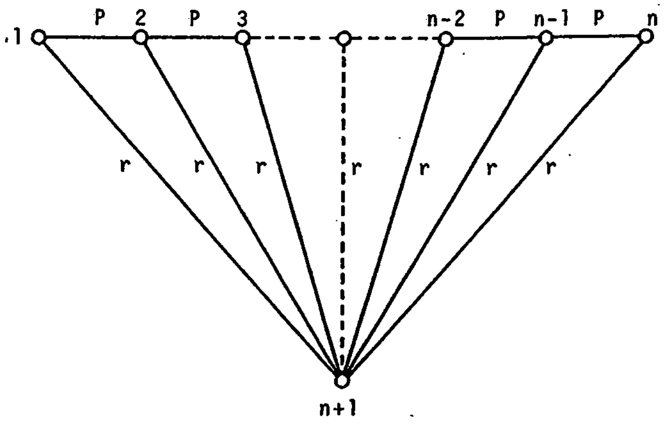

Figure 6: *Multigraph ladder N.*

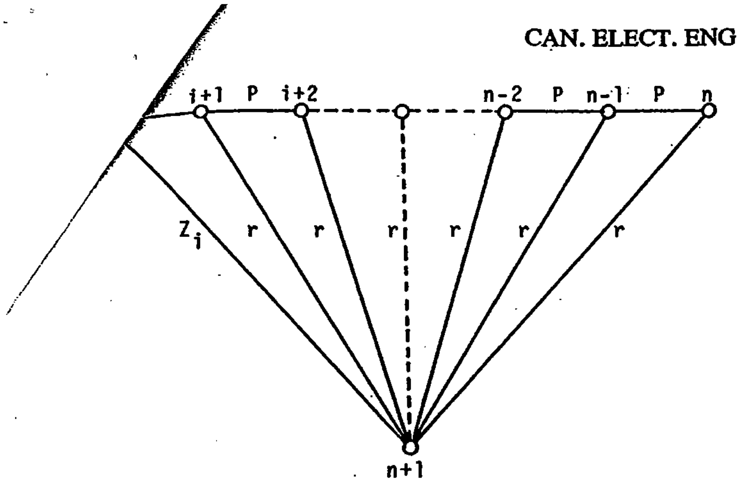

Figure 7: N°
$\{i, i+1, i+2, \ldots, n+1\}$.

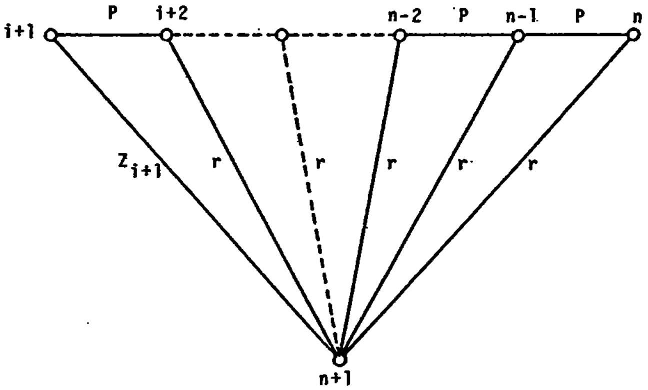

Figure 8: N°
$\{i+1, i+2, \ldots, n+1\}$.

If we choose $\Gamma_o = 0$, then Eq. (15) will become valid for $n = 2$.

We can solve Eq. (15) for Γ_n and obtain Γ_n as

$$\Gamma_n = \frac{r}{\alpha - \beta}(\alpha^n - \beta^n) \tag{17}$$

where

$$\alpha = \frac{(2P + r) + \sqrt{4Pr + r^2}}{2}$$

and

$$\beta = \frac{(2P + r) - \sqrt{4Pr + r^2}}{2}. \tag{18}$$

Note that $\alpha\beta = P^2$.

A recurrence relation between $\{W_n\}$ and $\{\Gamma_n\}$

Let W_n denote the number of spanning trees in an $(n + 1)$-node multigraph wheel in which P is the multiplicity of the rims and r is the multiplicity of the spokes. In reference 3 it is shown that

$$W_n = (2P + r)\,W_{n-1} - P^2W_{n-2} + 2P^{n-1}r, \; n \geq 3. \tag{19}$$

It is easy to see that

$$W_1 = r \text{ and } W_2 = 4Pr + r^2.$$

Choosing $W_o = 0$, we can see that Eq. (19) becomes valid for $n = 2$. Eq. (19) can be solved for W_n, which is given below:

$$W_n = \alpha^n + \beta^n - 2P^n \tag{20}$$

where α and β are as defined in Eq. (18).

The similarity between Eqs. (15) and (19), and between Eqs. (17) and (20) suggests that a relation connecting $\{W_n\}$ and $\{\Gamma_n\}$ may exist. This is indeed true and we establish this now.

Using Eq. (17), we obtain the following:

$$\Gamma_{n-1} - P^2\Gamma_{n-1} = \frac{r}{\alpha - \beta}[\alpha^{n+1} - \beta^{n+1} - P^2\alpha^{n-1} + P^2\beta^{n-1}]$$

$$= \frac{r}{\alpha - \beta}\left[\alpha^n\left(\alpha - \frac{P^2}{\alpha}\right) - \beta^n\left(\beta - \frac{P^2}{\beta}\right)\right]$$

$$= \frac{r}{\alpha - \beta}\left[\alpha^n(\alpha - \beta) - \beta^n(\beta - \alpha)\right]$$

$$= (\alpha^n + \beta^n)$$

$$= r(W_n + 2P^n). \tag{21}$$

Thus, we get from Eq. (21)

$$W_n = \frac{1}{r}(\Gamma_{n+1} - P^2\Gamma_{n-1}) - 2P^n. \tag{22}$$

The above relation is valid for all $n \geq 2$. This result also generalizes a similar result given in reference 2 for the special case of simple wheels and ladders.

Number of spanning trees in a cascade of 2-port networks

In reference 1, Myers obtained several interesting relations for counting spanning trees in a cascade of 2-port networks. In proving these results, he used some very ingenious but involved combinatorial arguments. In this section, we give a simple proof of one of these results (Eq. (11) in reference 1). All the other results in reference 1 can be proved in a similar manner. Our proof here is based on the approach used in the second section of this paper and some well-known network-theoretic results.

Consider a cascade C_n of n arbitrary 2-port networks $G_1, G_2, \ldots, G_n$ connected in that order as shown in Figure 9, with $(0,0')$ the input terminal pair of G_1 and (k, k') $k = 1,2,\ldots n$, the output terminal pair of G_k in C_n. We shall follow the same notation as in reference 1. Thus, we shall denote by g_k the graph of G_k and let

$n_k =$ an arbitrary spanning tree of g_k
$x_k =$ an arbitrary spanning 2-tree $T_{[k-1],[k-1]'}$ of g_k
$y_k =$ an arbitrary spanning 2-tree $T_{k,k'}$ of g_k
$Z_k =$ an arbitrary spanning 3-tree $T_{[k-1],k,[k-1]'k'}$;
$T_{[k-1],k',[k-1]'k}$;$T_{[k-1]k,[k-1]',k'}$; or $T_{[k-1]k',[k-1]',k}$ of g_k

With

$N_k =$ the sum of admittance-products of spanning trees n_k in g_k
$X_k =$ the sum of admittance-products of spanning 2-trees x_k in g_k
$Y_k =$ the sum of admittance-products of spanning 2-trees y_k in g_k
$Z_k =$ the sum of admittance-products of spanning 3-trees z_k in g_k

Note that X_k, Y_k and Z_k are respectively the sums of tree-admittance-products of the networks in Figures 10(a), (b) and (c).

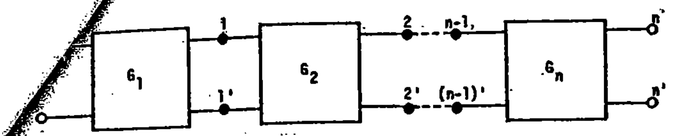

Figure 9: *C_n, cascade of n arbitrary 2-port networks $G_1, G_2, \ldots, G_n$.*

If C_k denotes the cascade of the first k 2-port networks $G_1, G_2, \ldots, G_k$, then let

$N_{1,2,\ldots,k}$ = the sum of admittance-products of spanning trees of the graph of C_k
$Y_{1,2,\ldots,k}$ = the sum of admittance-products of the spanning 2-trees $T_{k,k'}$ of the graph C_k.

Our problem is to find a recurrence relation relating $N_{1,2,\ldots,n}$, $N_{1,2,\ldots,n-1}$, $N_{1,2,\ldots,n-2}$, N_n, X_n, N_{n-1}, X_{n-1}, Y_{n-1} and Z_{n-1}. We shall first determine $N_{1,2,\ldots,n}$, using Eq. (1). To do so, we shall let K = the set of nodes of G_n except $(n-1)$ and $(n-1)'$. Then short-circuiting the nodes not in K, we get the network in Figure 11. As mentioned earlier, the sum of tree-admittance-products of this network is X_n.

Suppressing the nodes in K, we get the network in Figure 12 where D_{in} is the driving point admittance of G_n across the terminal pair $(n-1)$ and $(n-1)'$. It is known that[4]

$$D_{in} = \frac{N_n}{X_n}.$$

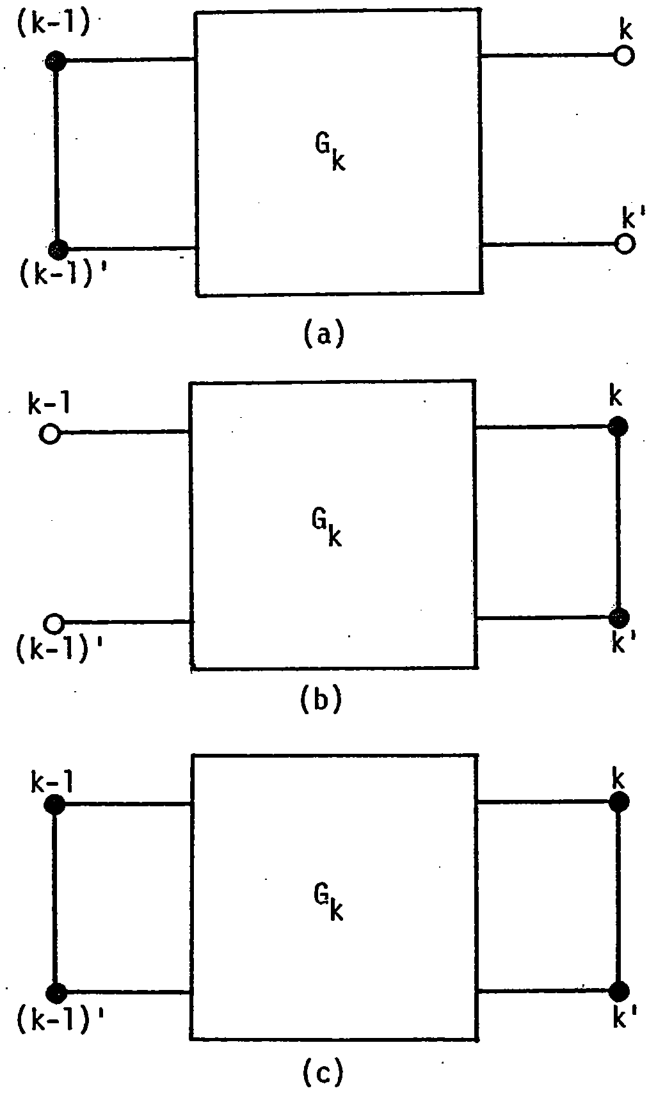

Figure 10: *Networks derived from G_k.*

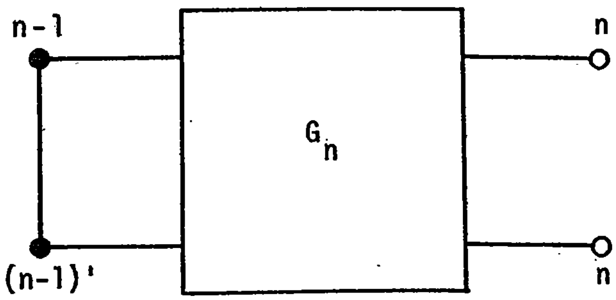

Figure 11: *Network derived from C_n by short-circuiting all the nodes not in G_n.*

Also, the sum W_1 of admittance-products of the spanning trees of the network in Figure 12 is given by

$$W_1 = N_{1,2,\ldots,n-1} + D_{in} \cdot Y_{1,2,\ldots,n-1}$$

since $N_{1,2,\ldots,n-1}$ is the sum of admittance-products of spanning trees of C_{n-1} and $Y_{1,2,\ldots,n-1}$ is the sum of the admittance-products of the spanning trees of the network obtained from C_{n-1} by short-circuiting the nodes $(n-1)$ and $(n-1)'$. So using Eq. (1), we get

$$\begin{aligned} N_{1,2,\ldots,n} &= X_n \cdot W_1 \\ &= X_n \cdot (N_{1,2,\ldots,n-1} + \frac{N_n}{X_n} Y_{1,2,\ldots,n-1}) \\ &= X_n \cdot N_{1,2,\ldots,n-1} + N_n \cdot Y_{1,2,\ldots,n-1} \qquad (23) \end{aligned}$$

Similarly,

$$N_{1,2,\ldots,n-1} = X_{n-1} \cdot N_{1,2,\ldots,n-2} + N_{n-1} \cdot Y_{1,2,\ldots,n-2}. \qquad (24)$$

If we could eliminate $Y_{1,2,\ldots,n-1}$ and $Y_{1,2,\ldots,n-2}$ from the above equations, we can obtain the desired recurrence relation. To do this, we shall evaluate $Y_{1,2,\ldots,n-1}$ using Eq. (1). By definition, $Y_{1,2,\ldots,n-1}$ is the sum of admittance-products of spanning trees of the network in Figure 13. If we again let K = the set of nodes of G_{n-1} except $(n-2)$ and $(n-2)'$, we can proceed as before and express $Y_{1,2,\ldots,n-1}$ in terms of the sums of admittance-products of the networks shown in Figures 14(a) and (b) as,

$$Y_{1,2,\ldots,n-1} = Z_{n-1} \cdot N_{1,2,\ldots,n-2} + Y_{n-1} \cdot Y_{1,2,\ldots,n-2} \qquad (25)$$

Using Eq. (25) in Eq. (23) and eliminating $Y_{1,2,\ldots,n-2}$ from the resulting equation as well as Eq. (25), we get the following.

$$\begin{aligned} N_{1,2,\ldots,n} N_{n-1} \\ &= (N_{n-1} X_n + Y_{n-1} N_n) N_{1,2,\ldots,n-1} + \\ &\qquad (N_{n-1} Z_{n-1} - X_{n-1} Y_{n-1}) N_{1,2,\ldots,n-2} N_n. \end{aligned}$$

The above is the same as Eq. (11) in reference 1.

All the other formulas derived in reference 1 can also be established using the approach given in the second section.

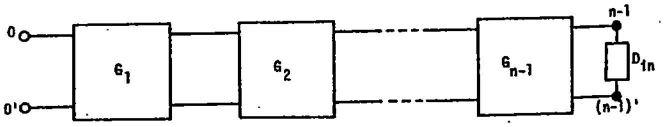

Figure 12: *Network derived from C_n by suppressing all the nodes of G_n except $(n-1)$ and $(n-1)'$.*

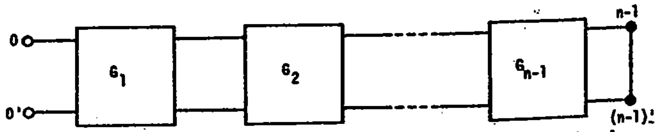

Figure 13: *Network derived from C_{n-1} by short-circuiting the nodes $(n-1)$ and $(n-1)'$.*

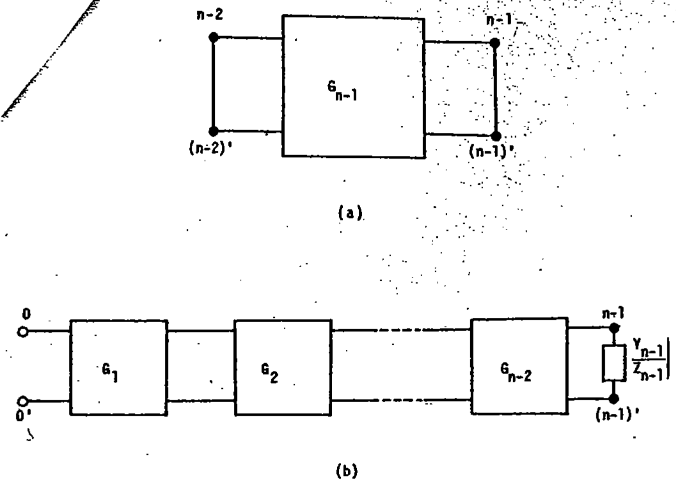

Figure 14: *(a) Network derived from the network of Figure 13 by short-circuiting all the nodes not in G_{n-1}; (b) Network derived from the network of Figure 13 by suppressing all the nodes in G_{n-1} except (n−2) and (n−2)'.*

Acknowledgement

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Conclusion

In this paper, we have given a network-theoretic approach, based on a theorem in the theory of determinants, for counting the number of spanning trees of a graph. We have examined the significance of this approach by presenting some new results and giving simpler proofs of some known results.[1-4] Though for purposes of illustration, we have established only one of the formulas in reference 1, all the other formulas given in that paper can also be established in a similar manner. We would like to add that this approach has proved very useful in a recent work on the complexity analysis of a spanning tree enumeration algorithm.[9] These results highlight the generality of the new approach as well as the insight it provides.

References

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