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# SYNTHESIS OF RATIONAL VOLTAGE TRANSFER MATRICES USING MINIMUM NUMBER OF CAPACITORS WITH OPERATIONAL AMPLIFIERS

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## Abstract

A simple and novel method of synthesizing any arbitrary nxn voltage transfer matrix T(s) of real rational functions of the complex frequency variable s, using the techniques of reactance extraction is given. The unique feature of the method is that of utilization of minimum number of capacitors, p, equal to the degree of the transfer matrix (using at most 2(n+p) operational amplifiers) with all the capacitors grounded.


## I. INTRODUCTION

Even though much progress has been made in developing synthesis techniques to realize any arbitrary nxn matrix Y(s) of real rational functions of the complex frequency variable s, as the short circuit admittance matrix of a n-port active RC network starting from Sandberg [1], [2] to Mann and Pike [6] and recently by Melvin and Bickart [7], little work has been done in developing systematic synthesis techniques to realize an arbitrary voltage transfer matrix barring a few to mention like that of Hilberman [5] and Kim and Su [8].

All the above configurations, except those of Mann and Pike's [6] and Melvin and Bickart's [7], [11], either to realize Y(s) or T(s) use a larger number of capacitors (with possibly some of them floating) than the minimum number required, which is equal to the degree [9] p of the given matrix, with minimum number of active devices. But with the advent of integrated technology, it has become necessary to use minimum number of capacitors, with one end grounded, even at the cost of active devices, to make the network performance reliable. Mann and Pike, bearing this in mind, using the techniques of reactance extraction showed that any arbitrary matrix Y(s) can be realized as the short circuit admittance of an n-port active RC network, with an increase in the number of controlled sources. Recently Melvin and Bickart [7] have elaborated on this and given a structure to realize Y(s).

The results of Mann & Pike [6] and Melvin and Bickart [7] suggest that a realization of T(s) with minimum number of capacitors may be possible, with an increase in the number of amplifiers and it has been proved here that it is always possible using at most 2(n+p) operational amplifiers. The same configuration proposed by Melvin and Bickart [7] for the subnetwork of capacitors is taken, thus retaining the advantages of that method namely all the capacitors are grounded. Also all the 2n-ports have the ground as a common terminal.

## II. SYNTHESIS PROCEDURE

The main results established in the paper can be given as a theorem as follows:

THEOREM:

Any nxn matrix T(s) of real rational functions of the complex frequency variable s can be realized as the voltage transfer matrix of a 2n-port active RC network containing the theoretical minimum number of capacitors p required, where p is the degree of T(s), and at most 2(n+p) operational amplifiers. All the capacitors and the 2n-ports will have the ground as a common terminal.

Note: If T(s) is a nxm (or mxn) matrix the number of operational amplifiers required will be a most n+m+2p.

As a proof for the above theorem we present a step-by-step realization procedure for T(s). We can take T(s) to be a nxn matrix without any loss in generality, the same procedure being valid for rectangular transfer matrices.

First let us assume T(s) is regular at infinity and consider the network block diagram given in Figure 1. If we denote the n vectors of input and output voltages as $e_1$ and $e_3$, respectively, and $e_2$, a p-state vector, application of Ho-Kalman algorithm [10] yields the F, G, H & J real constant matrices of a set of state equations.

Where

$$e_2 = F.\dot{e}_2 + G.e_1 \tag{1}$$

$$e_3 = H.\dot{e}_2 + J.e_1 \tag{2}$$

with

$$T(s) = J + H(SI-F)^{-1}G \tag{3}$$

We can identify $e_2$ as the voltage vector of capacitor subnetwork $N_c$. Then this establishes that p-capacitors are necessary and sufficient in any possible realization of T(s). The relationship imposed by $N_c$ on $e_2$ and $i_2$ is

$$i_2 = -C\dot{e}_2 \tag{4}$$

where C is a pxp matrix with positive diagonal entries only for our chosen configuration.

The subnetwork $N_R$ is assumed to be of the form shown in Figure 2, where $N_R$ is a 4n+3p+1 port common terminal resistive network. Its current voltage relationship being

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix} = \begin{bmatrix} g_{11} & g'_{21} & g_{13} & g_{14} & g_{15} & g_{16} & g'_{71} \\ g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} & g'_{72} \\ g'_{13} & g'_{23} & g_{33} & g_{34} & g_{35} & g_{36} & g'_{73} \\ g'_{14} & g'_{24} & g'_{34} & g_{44} & g_{45} & g_{46} & g'_{74} \\ g'_{15} & g'_{25} & g'_{35} & g'_{45} & g_{55} & g_{56} & g'_{75} \\ g'_{16} & g'_{26} & g'_{36} & g'_{46} & g'_{56} & g_{66} & g'_{76} \\ g_{71} & g_{72} & g_{73} & g_{74} & g_{75} & g_{76} & g_{77} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix} \tag{5}$$

$$= G \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix}$$

where $i_1, i_3, i_4, i_7, e_1, e_3, e_4$ and $e_7$ are n vectors, $i_2, i_5, i_6, e_2, e_5$ and $e_6$ are p vectors and the g's sub matrices, with primes denoting transposition. G, being the short circuit admittance matrix of common ground resistive network, has to be hyperdominant.

The introduction of active devices as shown in Figure 2 imposes the following constraints.

$$e_7 = i_7 = 0,\ e_6 = B_1 A_1 e_2,\ e_5 = A_1 e_2,\ e_4 = -e_1 \qquad (6)$$

$A_1$ = dia $(a_1, a_2, \ldots, a_p)$ $a_i \geq +1$ or $= 0$ for all i.

$B_1$ = dia $(b_1, b_2, \ldots, b_p)$ $b_i \leq 0$ for all i.

From (4) and (6), $i_7 = 0$ leads to (assuming $g_{72} = 0$) the expression

$$e_3 = -g_{73}^{-1}(g_{75}A_1 + g_{76}B_1A_1)e_2 + g_{73}^{-1}(g_{74} - g_{71})e_1 \qquad (7)$$

and since $i_2 = -C\dot{e}_2$ we get (with $g_{23} = 0$)

$$\dot{e}_2 = -C^{-1}(g_{22} + g_{25}A_1 + g_{26}A_1B_1)e_2 - C^{-1}(g_{21} - g_{24})e_1 \qquad (8)$$

Thus the problem of realization of T(s) has been reduced to that of identifying various terms of g-sub matrices of $N_R$ from (1), (7) and (8) subject to the condition that G is hyperdominant. (1), (7) and (8) lead to

$$\begin{aligned} H &= -g_{73}^{-1}(g_{75}A_1 + g_{76}B_1A_1) \ldots\ldots\ldots\ldots \text{i} \\ J &= g_{73}^{-1}(g_{74} - g_{71}) \ldots\ldots\ldots\ldots \text{ii} \\ F &= -C^{-1}(g_{22} + g_{25}A_1 + g_{26}B_1A_1) \ldots\ldots\ldots \text{iii} \\ G &= -C^{-1}(g_{21} - g_{24}) \ldots\ldots\ldots\ldots \text{iv} \end{aligned} \qquad (9)$$

Now we give a step-by-step procedure to identify various terms.

Step 1: From (9) iv, assuming suitable value for C find $g_{21}$ and $g_{24}$ such that they are sub matrices with non-positve entries.

Step 2: Since $g_{21}$ and $g_{24}$ are fixed, each diagonal entry in the sub matrix $g_{22}$ should be such that it is much greater than the modulus of sum of elements in the corresponding rows of $g_{21}$ and $g_{24}$. Thus we can assume $g_{22}$ to be a matrix of positive diagonal entries only, subject to the above condition.

Step 3: Then (9) iii yields

$-CF - g_{22} = g_{25}A_1 + g_{26}B_1A_1 = P + M$ where P contains all non-negative elements and M contains all non-positive elements. Hence, we get

$$g_{25}A_1 = M$$

$$g_{26}B_1A_1 = P$$

Suitable values for $B_1$ and $A_1$ are assumed and $g_{25}$ and $g_{26}$ are found.

Step 4: From (9) i and ii other terms are found by suitable assignment.

Step 5: Having found out the terms that have come into picture, we can complete the remaining entires in the G matrix arbitrarily taking care to see that it is hyperdominant.

The above mentioned steps demonstrate clearly that a decomposition is always possible, hence the validity of the theorem.

Example:

Let us take the example given by Melvin and Bickart for Y(s) and realize it as a voltage transfer matrix of a 2n-port network.

$$T(s) = \begin{bmatrix} \frac{2s+1}{s+1} & \frac{s}{s+1} \\ \frac{s+\frac{1}{2}}{s+1} & \frac{2s-1}{s+1} \end{bmatrix}$$

A set of (F,G,H,J) matrices of State Equations are

$$F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} -1 & -1 \\ -1/2 & -3 \end{bmatrix} \text{ and } J = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Let us assume

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } g_{73} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then from (9) iv we get

$$g_{21} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } g_{24} = 0$$

Let us fix $g_{22} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$

(9) iii then gives

$$g_{25}A_1 = \begin{bmatrix} -9 & 0 \\ 0 & -9 \end{bmatrix} \text{ and } g_{26}B_1A_1 = 0$$

Hence we can take

$$g_{25} = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix};\ A_1 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \text{ and } g_{26} = 0$$

Now (9) ii gives

$$g_{74} = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \text{ and } g_{71} = 0$$

and (9) i gives

$$g_{76}B_1A_1 = 0 \text{ and } g_{75}A_1 = \begin{bmatrix} -1 & -1 \\ -1/2 & -3 \end{bmatrix}$$

Now we can complete the G matrix, taking care to see that it is hyperdominant. The G matrix arrived at being

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & -1/3 & -1/6 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & -1/3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -2 & -1 & -1/3 & -1/3 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -2 & -1/6 & -1 & 0 & 0 & 0 & 6 \end{bmatrix}$$

Having found the G matrix of $N_R$ the network can easily be constructed.

Note: The number of operational amplifiers used is 6, less than 2(n+p) since we have got one matrix $B_1$ to be zero.

## III. CONCLUSIONS

We have given above a systematic synthesis procedure for realizing any arbitrary voltage transfer matrix of real rational functions of the complex frequency variable s. The unique features of the method are;

(i) all the capacitances are grounded, an attractive thing from the point of view of integrated circuit realization, and

(ii) all the ports have one terminal as the common ground. Even though the reduction of number of capacitors is effected by an increase in the number of active devices, this solution will be attractive for realizing matrices of higher order and lower degree.

When compared to Su's [6] method of realizing voltage transfer matrices, this method utilizes the minimum number of capacitors, all being grounded, with a slight increase in the number of resistors used, at the cost of active devices. With the advent of integrated circuits, it is rather welcome to reduce the number of passive elements even if it results in an increase in the number of active devices. Also when the transfer voltage matrix is rectangular one (nxm), the method presented here requires a maximum of m+n+2p active devices whereas Su's [6] method requires 2n or 2m active devices whichever is greater. This means that the method may be utilizing a few number of active devices than Su's method when used to realize a rectangular transfer matrix with the degree much lower than the order of the matrix in addition to the fact that the number of capacitors used are minimum. The saving in the number of active devices may be still more when some entries in $A_1$ or $B_1$ becomes zero, as in the example given.

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## BIOGRAPHY

P.A. Ramamoorthy was born in Tindivanam, Tamilnadu, India on December 20, 1949. He received his B.E. degree in Electrical Engineering from the University of Madras in 1971. He joined the Indian Institute of Technology, Madras in 1971 and is working towards his M.S. degree at the Indian Institute of Technology, Madras, India.

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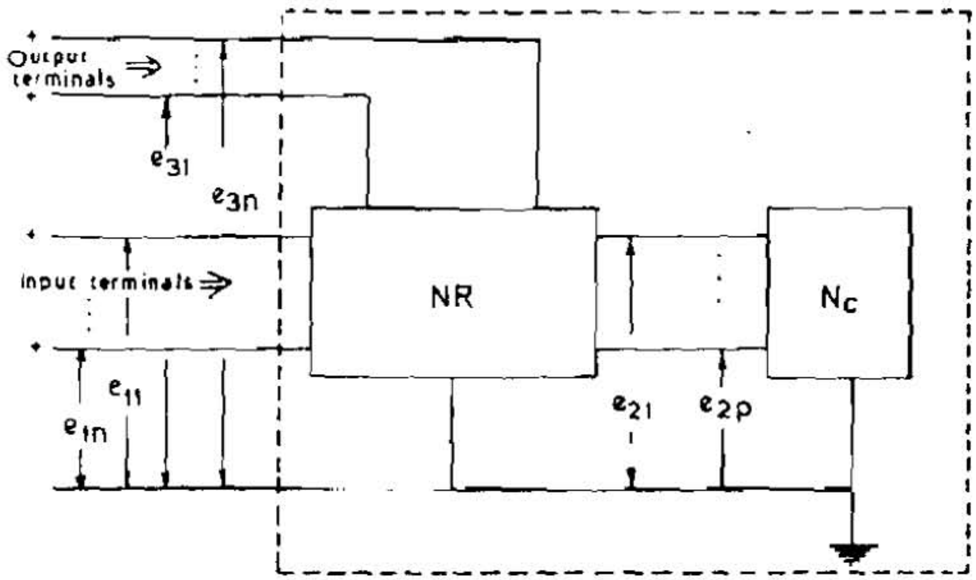


Fig 1 Network Block Diagram

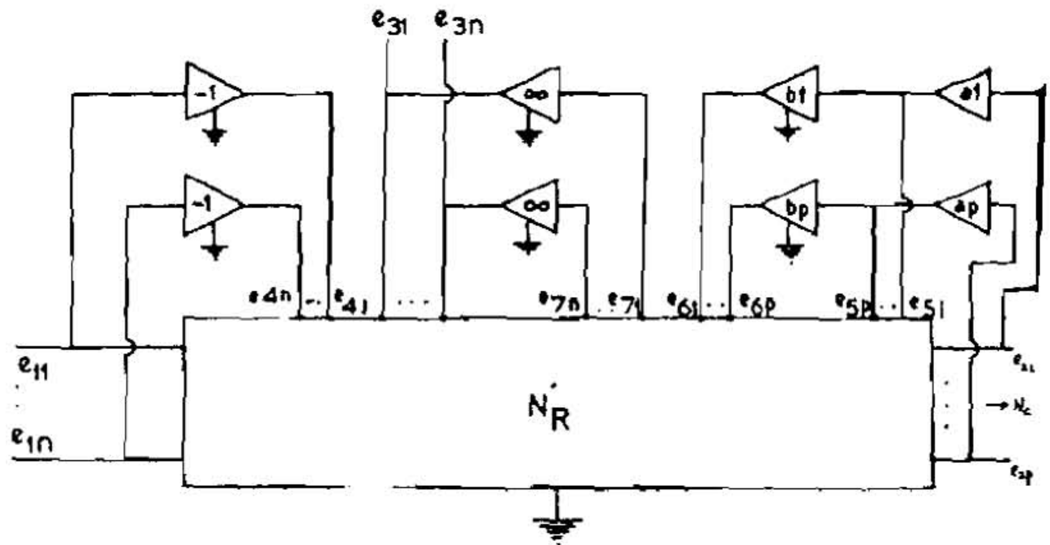


Fig 2 Subnetwork $N_R$