Synthesis applications of the modified cut-set matrix

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Synopsis

A procedure for the realisation of a general *K*network, which is a generalisation of the conventional $2n$-node network realisation of a dominant matrix, is given. It is shown that a set of dominant matrices can be realised by $2n$-node n-port networks having the same modified cut-set matrix. This technique is used in evolving a procedure for the realisation of 2-element-kind n-port networks.

1 Introduction

It is well known[1] that a real symmetric dominant matrix of order n can be realised as the short-circuit conductance matrix of a $2n$-node n-port resistive network which contains a pair of equal conductances interconnecting the terminals of every pair of ports. Recently, a generalisation of this procedure was proposed by the authors,[2] and it is shown that such a matrix can be realised as the short-circuit conductance matrix of one of a class of $2n$-node n-port networks called *K*networks. Realisation procedures covering a subclass of *K*networks, called constant *K*networks, have also been given.

In a companion paper,[3] the properties of the modified cut-set matrix and the usefulness of the modified cut-set matrix in formulating a new criterion for the proper parallel connection of n-port networks and in establishing a general procedure for the generation of a class of continuously equivalent networks for a given n-port network have been discussed. In this paper, the results obtained in Reference 3 are used to evolve a general procedure for the synthesis of resistive *K*networks. It is also shown that this procedure can be extended to obtain a large class of realisations for a given short-circuit admittance matrix of a 2element-kind n-port network when the residue matrices of the given Y matrix are dominant. If the residue matrices are not dominant, the transfer admittances can be realised, within a multiplicative constant.

2 *K*networks and their properties

In this Section, the term *K*network is defined, and certain properties of *K*networks already reported in Reference 2 are summarised.

Consider an n-port network with $2n$ nodes. Let port i be excited with a source of unit voltage and let all the other ports be short-circuited. Then the potential factor K_{ij} refers to the potential of the positive reference terminal of port j with respect to the negative reference terminal of port i. K_{ii} is therefore unity. A $2n$-node n-port network in which the relationship $K_{i1} = K_{i2} = \ldots = K_{ir} \ldots = K_{in}$, $r \neq i$, holds for every i is termed a *K*network. The symbol K_i used to refer to this common value is termed the potential factor of port i in the *K*network.

Let the network configuration between any two ports i and j be as shown in Fig. 1, where a_{ij}, b_{ij}, c_{ij}, d_{ij}, g_i and g_j, refer to the conductances of the respective edges. That all these values are finite is implicit in our assumption of a $2n$-node network.

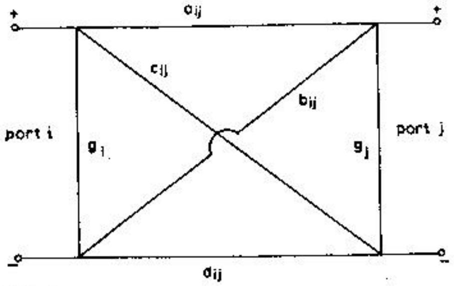

Fig. 1
Conductances of the edges interconnecting ports i and j

In a *K*network with port i excited and all the other ports short-circuited, the short-circuited ports are at the same potential, and the edges interconnecting the short-circuited ports do not carry current. Hence the transfer admittance y_{ij} between ports i and j depends only on the edges interconnecting the ports i and j, and it is given by

$$\begin{aligned} y_{ij} &= c_{ij}(1 - K_i) - K_i d_{ij} \\ &= K_i b_{ij} - a_{ij}(1 - K_i) \\ &= \frac{b_{ij}c_{ij} - a_{ij}d_{ij}}{a_{ij} + b_{ij} + c_{ij} + d_{ij}} \end{aligned} \quad (1)$$

y_{ii}, the driving-point admittance of port i, is obtained from

$$y_{ii} = \sum_{\substack{j=1 \\ j \neq i}}^{n} (y_{ii})_j + g_i$$

$$\text{where } (y_{ii})_j = \frac{(a_{ij} + c_{ij})(b_{ij} + d_{ij})}{a_{ij} + b_{ij} + c_{ij} + d_{ij}}$$

$$= K_i(b_{ij} + d_{ij})$$

$$= (a_{ij} + c_{ij})(1 - K_i) \quad (2)$$

$(y_{ii})_j$ may be considered as the contribution to y_{ii} of the conductances of the edges interconnecting the ports i and j.

Certain important properties of *K*networks are as follows.

Property 1. The potential factor K_i satisfies the inequality $0 \leqslant K_i \leqslant 1$ for a network containing only nonnegative conductances.

Property 2. When two *K*networks N_1 and N_2, with the same set of potential factors, i.e. the value of K_i is the same in N_1 and N_2 for every port i, and having Y_1 and Y_2 as the short-circuit admittance matrices, are connected in parallel:

(*a*) the resulting network is a *K*network and the potential factor of every one of its ports is the same as that of the corresponding port in either of the constituent networks

(*b*) the short-circuit admittance matrix Y of the resulting network is given by $Y = Y_1 + Y_2$.

Property 3. The short-circuit conductance matrix of a $2n$-node *K*network containing only nonnegative conductances is dominant.

A *K*network in which the potential factors of all ports are equal is called a constant *K*network. For such a network, $b_{ij} = c_{ij}$ for all i and j, since $K_i = K_j$.

3 Realisation of a 2 × 2 real dominant matrix

In this Section, the problem of realisation of a 2 × 2 dominant short-circuit conductance matrix by a 2-port network having a specified modified cut-set matrix is considered.

Consider the 2-port resistive network shown in Fig. 2, where a, b, c, d, g_1 and g_2 refer to the conductances of the respective edges.

Choosing the tree comprising the edges $e_{1',1}$, $e_{1,2'}$, $e_{2',2}$, the fundamental cut-set matrix C_0 is partitioned as

$$C_0 = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

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where the rows of C_1 correspond to the port edges $e_{1',1}$ and $e_{2',2}$ and the row of C_2 corresponds to the nonport edge $e_{1,2'}$:

$$C_0 = \begin{bmatrix} C_1 \\ \hline C_2 \end{bmatrix}$$

$$= \begin{array}{c} \begin{matrix} e_{1',1} & e_{1',2'} & e_{1',2} & e_{1,2'} & e_{1,2} & e_{2',2} \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \quad . \quad . \quad (3)$$

If K_1 and K_2 are the potential factors of ports 1 and 2, respectively, the modified cut-set matrix of the 2-port network

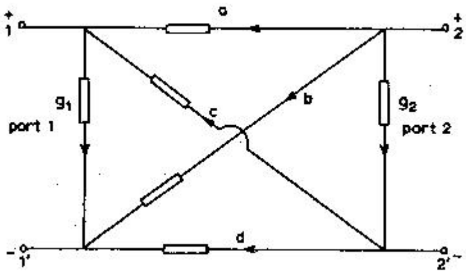

Fig. 2
2-port resistive network

for the edge and port orientations shown in Fig. 2 can be written[3] as

$$C = \begin{array}{c} \begin{matrix} e_{1',1} & e_{1',2'} & e_{1',2} & e_{1,2'} & e_{1,2} & e_{2',2} \end{matrix} \\ \begin{bmatrix} 1 & K_1 & K_1 & K_1 - 1 & K_1 - 1 & 0 \\ 0 & -K_2 & 1 - K_2 & -K_2 & 1 - K_2 & 1 \end{bmatrix} \end{array} \quad . \quad (4)$$

Let it be required to realise the following real symmetric dominant matrix:

$$Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

as the short-circuit conductance matrix of a 2-port network having the form shown in Fig. 2.

Let

$$G = \begin{bmatrix} g_1 & & & & & \\ & d & & & 0 & \\ & & b & & & \\ & & & c & & \\ & 0 & & & a & \\ & & & & & g_2 \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

the edge conductance matrix, satisfy the two equations

$$C \; G \; C_1' = Y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and $$C \; G \; C_2' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This implies[3] that the modified cut-set matrix and the short-circuit conductance matrix of the 2-port network having G as its edge conductance matrix are equal to C and Y, respectively. Eqns. 7 and 8 can be together put in the following form:

$$\begin{bmatrix} 1 & K_1 & K_1 & 0 & 0 & 0 \\ 0 & 0 & 1 - K_2 & 0 & 1 - K_2 & 1 \\ 0 & 0 & K_1 & 0 & K_1 - 1 & 0 \\ 0 & -K_2 & 1 - K_2 & 0 & 0 & 0 \\ 0 & K_1 & K_1 & K_1 - 1 & K_1 - 1 & 0 \\ 0 & -K_2 & 1 - K_2 & -K_2 & 1 - K_2 & 0 \end{bmatrix} \begin{bmatrix} g_1 \\ d \\ b \\ c \\ a \\ g_2 \end{bmatrix} = \begin{bmatrix} y_{11} \\ y_{22} \\ y_{12} \\ y_{12} \\ 0 \\ 0 \end{bmatrix} \quad (9)$$

Given the short-circuit conductance matrix Y and the modified cut-set matrix C, a unique solution of the six conductances does not exist, since the coefficient matrix in these equations has a rank of five. We can, however, choose one of the conductances, say b, as an arbitrary parameter, and express the other conductances g_1, g_2, a, c and d in terms of b, K_1, K_2, y_{11}, y_{12} and y_{22}, as follows:

$$d = \frac{1 - K_2}{K_2} b - \frac{y_{12}}{K_2}$$

$$a = \frac{K_1 b}{1 - K_1} - \frac{y_{12}}{1 - K_1}$$

$$c = \frac{K_1(1 - K_2)}{K_2(1 - K_1)} b - \frac{K_1 - K_2}{(1 - K_1)K_2} y_{12}$$

$$g_1 = y_{11} + \frac{K_1}{K_2} y_{12} - \frac{K_1}{K_2} b$$

$$g_2 = y_{22} + \frac{1 - K_2}{1 - K_1} y_{12} - \frac{1 - K_2}{1 - K_1} b \quad . \quad . \quad . \quad . \quad (10)$$

For a proper realisation, it is required that g_1, g_2, d, c and a be nonnegative for a nonnegative b. This restriction leads to the following inequalities, obtained by requiring that the right-hand side of eqn. 10 be nonnegative:

$$\left. \begin{aligned} & b \geqslant \frac{y_{12}}{1 - K_2} \\ & b \geqslant \frac{y_{12}}{K_1} \\ & b \geqslant \frac{y_{12}(K_1 - K_2)}{(1 - K_2)K_1} \\ & b \leqslant y_{11} \frac{K_2}{K_1} + y_{12} \\ & b \leqslant y_{22} \frac{1 - K_1}{1 - K_2} + y_{12} \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If K_1 and K_2, or, in other words, C, is specified, choosing a nonnegative b satisfying these inequalities and using eqn. 10, a proper 2-port realisation of Y having the specified modified cut-set matrix C and the configuration shown in Fig. 2 can be obtained.

If it is not possible to choose a nonnegative b satisfying the inequalities in expr. 11, it only means that the given dominant matrix Y cannot be realised with the specified modified cut-set matrix. However, it can be realised with some other suitably chosen modified cut-set matrix.

We now consider the question of determination of an appropriate modified cut-set matrix which leads to a proper realisation of a given Y in the form of the network shown in Fig. 2. One solution of this problem is adequate for our purpose. It is shown in Appendix 8 that, if K_1 and K_2 are chosen to satisfy the following restrictions, the requirements of eqn. 11 are met, and a proper realisation is possible:

Case (a): $y_{11} \geqslant y_{22}$

Choose $K_1 \geqslant K_2 \geqslant \frac{1}{2}$, so that

$$\frac{y_{22}}{|y_{12}|} \geqslant \frac{K_2}{1 - K_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Case (b): $y_{22} \geqslant y_{11}$

Choose $\frac{1}{2} \geqslant K_1 \geqslant K_2$, so that

$$\frac{y_{11}}{|y_{12}|} \geqslant \frac{1 - K_1}{K_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Depending on the relative magnitudes of y_{11} and y_{22} in a given Y, K_1 and K_2 may be chosen to satisfy either eqn. 12 or eqn. 13 to ensure a proper realisation of the matrix Y. Since $y_{22}/|y_{12}| \geqslant 1$ and $y_{11}/|y_{12}| \geqslant 1$, it is always possible to do this. It must be noted that there exists, in the general case, a large number of sets of K_1 and K_2 satisfying eqns. 12 or 13, and hence a large number of realisations for a given Y are possible.

The ideas presented here will be used in the Section 4 to obtain a procedure for the synthesis of general Knetworks.

4 Synthesis of general *K* networks

Definition 1. A real symmetric matrix $Y = [y_{ij}]$ is said to be marginally dominant if $y_{ii} = \sum_{\substack{j=1 \\ j \neq i}}^{n} |y_{ij}|$ for every i.

Definition 2. A real symmetric matrix $Y = [y_{ij}]$ is said to be superdominant if $y_{ii} > \sum_{\substack{j=1 \\ j \neq i}}^{n} |y_{ij}|$ for every i.

We now consider the problem of realisation of a real symmetric superdominant matrix Y as the short-circuit admittance matrix of a Knetwork in which no two potential factors are equal. Let

$$\Delta_i = y_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} |y_{ij}| \qquad (14)$$

and let

$$\Delta_i \geqslant \Delta_j \text{ if } j > i \qquad (15)$$

The condition in eqn. 15 involves no loss of generality, since any given matrix can be made to satisfy this requirement by a simple interchange of rows and corresponding columns, and such an interchange results only in a change in the ordering of the ports.

It may be recalled that a Knetwork has the property that, when any port i is excited with a voltage source and all the other ports short-circuited, the edges interconnecting the short-circuited ports do not carry any current. Hence the transfer admittance y_{ij} depends solely on the conductances of the edges interconnecting the ports i and j, and can be realised by viewing ports i and j as constituting a 2-port network. To this end, we consider the 2-port dominant short-circuit admittance matrix:

$$\begin{bmatrix} (y_i)_j & y_{ij} \\ y_{ij} & (y_j)_i \end{bmatrix} \qquad (16)$$

where $(y_i)_j = |y_{ij}| + \dfrac{\Delta_i}{n-1}$ (17)

and $(y_j)_i = |y_{ij}| + \dfrac{\Delta_j}{n-1}$

The realisation of this matrix can be effected by the methods of Section 3 after an appropriate choice of potential factors K_i and K_j. This results in a lattice structure between ports i and j and two conductances $(g_i)_j$ and $(g_j)_i$ shunting ports i and j, respectively. If the 2-port structure between every pair of ports is thus realised, it is easy to see that the overall n-port network has the specified short-circuit admittance matrix, since

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} (y_i)_j = (n-1)\frac{\Delta_i}{n-1} + \sum_{\substack{j=1 \\ j \neq i}}^{n} |y_{ij}| = y_{ii} \qquad (18)$$

The conductance in the final network-shunting port i is given by

$$g_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} (g_i)_j \qquad (19)$$

The problem of the realisation of the given Ymatrix thus reduces to the problem of realisation of a set of real symmetric matrices of order 2. It only remains to consider the choice of potential factors for the ports, such that they are all different and such that they are appropriate to the proper realisation of every 2-port conductance matrix of the type in eqn. 16. The following discussion is directed towards evolving a suitable procedure for this choice.

Let ϵ be the minimum of the set of values

$$\left\{\frac{\Delta_j}{(n-1)|y_{ij}|}\right\}$$

for all i and j with $j > i$. We then have, for all i and j with $j > i$,

$$\frac{(y_j)_i}{|y_{ij}|} = \frac{|y_{ij}| + \dfrac{\Delta_j}{(n-1)}}{|y_{ij}|} = 1 + \frac{\Delta_j}{(n-1)|y_{ij}|} \geqslant 1 + \epsilon \qquad (20)$$

If we choose K_1 so that

$$\frac{1+\epsilon}{2+\epsilon} \leqslant K_1 < \frac{1+2\epsilon}{2+2\epsilon} \qquad (21)$$

we ensure that

$$\tfrac{1}{2} < (1+\epsilon)(1-K_1) \leqslant K_1 \qquad (22)$$

If $K_2, K_3, \ldots, K_n$ are next chosen so that

$$\tfrac{1}{2} < K_n < K_{n-1} < K_{n-2} \ldots K_2 < (1+\epsilon)(1-K_1) \qquad (23)$$

we proceed to show that the choice meets our requirements.

First, in the realisation of the 2-port structure interconnecting port 1 and any other port i, we have $(y_1)_i \geqslant (y_i)_1$, since $\Delta_1 \geqslant \Delta_i$ from eqn. 15. For this 2-port realisation, case (*a*) of Section 3, and hence eqn. 12, is applicable. From eqns. 20 and 23, we have

$$\frac{(y_i)_1}{|y_{i1}|} \geqslant 1 + \epsilon \geqslant \frac{K_i}{1-K_i} \qquad (24)$$

and $K_1 > K_i > \frac{1}{2}$, satisfying the requirements of eqn. 12.

Secondly, for the realisation of the 2-port structure interconnecting ports i and j, $j > i$, we have $(y_i)_j \geqslant (y_j)_i$, since $\Delta_i \geqslant \Delta_j$ from eqn. 15. Again, case (*a*) of Section 3 is applicable. It is seen that the pertinent condition of eqn. 12 is satisfied:

$$\frac{(y_j)_i}{|y_{ij}|} \geqslant 1 + \epsilon \geqslant \frac{K_j}{1-K_1} \geqslant \frac{K_j}{1-K_i} \qquad (25)$$

Hence the choice of potential factors, as in eqns. 21 and 23, is appropriate for the proper realisation of the n-port network.

The steps in the procedure for the synthesis of general Knetworks can be summarised as follows:

(*a*) Rearrange the rows and the corresponding columns of the given matrix so that $\Delta_i \geqslant \Delta_j$ for $j > i$. Let the matrix after such a rearrangement be called the Ymatrix.

(*b*) Determine

$$\epsilon = \min\left\{\frac{\Delta_j}{(n-1)|y_{ij}|}\right\}, j > i$$

Choose

$$\frac{1+\epsilon}{2+\epsilon} \leqslant K_1 < \frac{1+2\epsilon}{2+2\epsilon}$$

and $\tfrac{1}{2} < K_n < K_{n-1} \ldots K_3 < K_2 < (1+\epsilon)(1-K_1)$

(*c*) Realise the 2-port short-circuit conductance matrix relative to ports i and j, $j > i$, by the network shown in Fig. 2.

(i) Choose a nonnegative b_{ij}, so that

$$\frac{y_{ij}}{1-K_j} \leqslant b_{ij} \leqslant \frac{|y_{ij}| + \dfrac{\Delta_j}{n-1}}{1-K_j}(1-K_i) + y_{ij}$$

(ii) Determine the conductances of the other elements as follows:

$$a_{ij} = \frac{K_i}{1-K_i} b_{ij} - \frac{y_{ij}}{1-K_i}$$

$$c_{ij} = \frac{K_i(1-K_j)}{K_j(1-K_i)} b_{ij} - \frac{y_{ij}(K_i - K_j)}{(1-K_i)K_j}$$

$$d_{ij} = \frac{(1-K_j)}{(K_j)} b_{ij} - \frac{y_{ij}}{K_j}$$

$$(g_i)_j = |y_{ij}| + \frac{\Delta_i}{(n-1)} + \frac{K_i}{K_j} y_{ij} - \frac{K_i}{K_j} b_{ij}$$

$$(g_j)_i = |y_{ij}| + \frac{\Delta_j}{(n-1)} + \frac{(1-K_j)}{(1-K_i)} y_{ij} - \frac{(1-K_j)}{(1-K_i)} b_{ij} \qquad (26)$$

(*d*) Repeat the procedure in step (*c*) for all pairs of ports and form the overall n-port network in which

$$g_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} (g_i)_j \qquad (27)$$

This completes the procedure for the synthesis of a superdominant real matrix by a *K*network in which potential factors may be chosen so that no two are equal. If the *Y*matrix is marginally dominant, i.e. $\epsilon = 0$, the choice is limited to the conventional realisation of a dominant matrix, in which, of course, all potential factors are equal to 1/2.

Example 1

We consider the realisation of the following short-circuit admittance matrix Y by a 4-port *K*network:

$$Y = \begin{bmatrix} 23 & -5 & 6 & 2 \\ -5 & 21 & -3 & 4 \\ 6 & -3 & 19 & 1 \\ 2 & 4 & 1 & 13 \end{bmatrix}$$

whence $\Delta_1 = 10$, $\Delta_2 = 9$, $\Delta_3 = 9$ and $\Delta_4 = 6$, and

$$\frac{\Delta_2}{3|y_{12}|} = \frac{9}{15}$$

$$\frac{\Delta_3}{3|y_{13}|} = \frac{9}{18}, \frac{\Delta_3}{3|y_{23}|} = \frac{9}{9}$$

$$\frac{\Delta_4}{3|y_{14}|} = \frac{6}{6}, \frac{\Delta_4}{3|y_{24}|} = \frac{6}{12}, \frac{\Delta_4}{3|y_{34}|} = \frac{6}{3}$$

and $\epsilon = \min\left\{\frac{\Delta_j}{3|y_{ij}|}\right\}$, for $j > i$, is $0\cdot5$.

Choose

$$\frac{1+\epsilon}{2+\epsilon} \leqslant K_1 < \frac{1+2\epsilon}{2+2\epsilon}$$

i.e. $$0\cdot6 \leqslant K_1 < \frac{2}{3}$$

Therefore choose $K_1 = 0\cdot6$, $(1+\epsilon)(1-K_1) = 0\cdot6$, $K_2 = 0\cdot58$, $K_3 = 0\cdot55$ and $K_4 = 0\cdot52$.

The choice of b_{ij} is governed by

$$\frac{y_{ij}}{1-K_j} \leqslant b_{ij} \leqslant \frac{(y_j)_i(1-K_i)}{(1-K_j)} + y_{ij}$$

Applying this criterion, the following values of the b parameters are chosen; all the conductances are in siemens:

$$b_{12} = 2;\ b_{13} = 13\cdot5;\ b_{23} = 2;\ b_{14} = 5;\ b_{24} = 3;\ b_{34} = 3$$

The other conductances are obtained using eqn. 26 and are listed in Table 1.

Table 1

EDGE CONDUCTANCES OF THE NETWORK REALISED IN EXAMPLE 1

i	j	a_{ij}	b_{ij}	c_{ij}	d_{ij}	$(g_i)_j$	$(g_j)_i$
1	2	$\frac{31}{2}$	2	$\frac{151}{58}$	$\frac{292}{29}$	$\frac{95}{87}$	$\frac{65}{100}$
1	3	$\frac{21}{4}$	$\frac{27}{2}$	$\frac{669}{44}$	$\frac{3}{22}$	$\frac{38}{33}$	$\frac{9}{16}$
1	4	$\frac{5}{2}$	5	$\frac{80}{13}$	$\frac{10}{13}$	$\frac{73}{39}$	$\frac{4}{10}$
2	3	$\frac{208}{21}$	2	$\frac{204}{77}$	$\frac{390}{55}$	$\frac{8}{11}$	$\frac{27}{42}$
2	4	$\frac{61}{21}$	9	$\frac{2832}{273}$	$\frac{8}{13}$	$\frac{37}{26}$	$\frac{2}{7}$
3	4	$\frac{13}{9}$	3	$\frac{127}{39}$	$\frac{11}{13}$	$\frac{49}{26}$	$\frac{13}{15}$

Conductances of the edges shunting the ports are obtained using eqn. 27, and are

$$g_1 = 4\frac{478}{4147}, g_2 = 2\frac{2289}{2860}, g_3 = 3\frac{393}{4368}, g_4 = 1\frac{58}{105}$$

Realisation of a superdominant short-circuit conductance matrix by a *K*network offers flexibility in the choice of potential factors, and hence a range of equivalent networks with different sets of potential factors can be obtained. Also, once a *K*network realising a given Y is obtained, a range of continuously equivalent networks with the same set of potential factors can be obtained using the simple procedure given in Reference 2. Thus it is possible to force some of the edge conductances to have arbitrarily preassigned values. In particular, some $n-1$ of the conductances shunting the ports may be reduced to zero.

5 Synthesis of 2-element-kind *n*-port networks

It has been suggested in literature[4, 5] that the realisation of the short-circuit admittance matrix Y of an *RLC* *n*-port network can be carried out by realising the residue matrices or the parameter matrices of the given Y matrix, and connecting in parallel the appropriate networks, realising the component matrices of Y. To prove successful, this approach requires that the parallel combination of the component networks be proper. In the light of the result obtained[3, 6] on the necessary and sufficient condition for the proper parallel combination of a set of *n*-port networks, realisation of the Y matrix of an *RCL* network by this approach leads to the problem of synthesis of a set of resistive *n*-port networks having the same modified cut-set matrix.

Several methods are available for realising resistive *n*-port networks with $n+1$ nodes, and, for such networks, the fundamental cut-set matrix with respect to the tree constituted by the port edges is the same as the modified cut-set matrix. Hence, synthesis of *RLC* *n*-port networks with $n+1$ nodes can be easily carried out if the same port configuration is maintained in the realisation of the component matrices. However, the synthesis of *RLC* *n*-port networks with more than $n+1$ nodes is not as simple as this. This has led Guillemin to remark:[7]

> Although of some academic interest, this more general problem (2*n*-node network realisation) does not have the practical significance that attaches to $(n+1)$-node realisation method. As is pointed out in another paper,[2] the latter technique is not merely a method for the synthesis of single-element-kind network, but is readily extended to the synthesis of *RLC* networks through appropriate interconnection of separate single-element-kind realisations. . . . A 2*n*-node realisation procedure, although possessing greater realisation potential for a single matrix, has no value in the more general multi-element-kind synthesis problem.

However, in this Section we proceed to show that we are not completely helpless (as surmised in the quotation) in the synthesis of *RLC* *n*-port networks with more than $n+1$ nodes. Since it is always possible to realise a set of real dominant matrices by *K*networks having the same modified cut-set matrix, the task can be easily accomplished if each residue matrix is real and dominant. Furthermore, when all the matrices are superdominant, a number of different sets of potential factors can always be found. We now consider the application of this method to synthesis of 2-element-kind *n*-port networks, since, for this class of networks, real residue matrices can be conveniently set up. It must be noted that, when the residue matrices are not real, the parameter matrices for a given *Y*matrix should be obtained. The same methods can be used for the realisation of the parameter matrices, but determining the parameter matrices is not always easy.

Let it be required to realise a set of *n*th-order real superdominant matrices $Y_1, Y_2, \ldots Y_p$ by *K*networks having the same modified cut-set matrix. In the discussion that follows, a quantity pertinent to matrix Y_k is distinguished by the superscript (k). It is assumed that, for each matrix Y_k,

$$\Delta_i^{(k)} \geqslant \Delta_j^{(k)} \text{ for all } i \text{ and } j (j > i) \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Let ϵ_k be the minimum of the set of values

$$\left\{\frac{\Delta_j^{(k)}}{(n-1)|y_{ij}^{(k)}|}\right\}$$

for all i and j $(j > i)$, and let ϵ_{min} be the minimum of the set of values $\{\epsilon_k\}$ for all $k = 1, \ldots, p$. It is clear that the set of potential factors $K_1, K_2 \ldots, K_n$ chosen to satisfy the following inequalities meet, for each matrix Y_k, the requirements of eqn. 12, and hence it forms an appropriate set of potential factors for the proper realisation of all the given super-dominant matrices:

$$\frac{1+\epsilon_{min}}{2+\epsilon_{min}} \leqslant K_1 < \frac{1+2\epsilon_{min}}{2+2\epsilon_{min}}$$

$$\tfrac{1}{2} < K_n < K_{n-1} \ldots K_2 < (1+\epsilon_{min})(1-K_1) \quad . \quad (29)$$

If it is not possible to arrange the rows and columns of the matrices as given, to satisfy the requirement of eqn. 28, we can always split up a matrix Y_k as $Y_k' + Y_k''$, where Y_k'' is a diagonal matrix having positive entries y_{ii}'' along the diagonal and Y_k' is a suitable superdominant matrix satisfying the requirement of eqn. 28. We then consider the set of matrices Y_k' for the purposes of determining an appropriate set of potential factors. After a network realising Y_k' is found, the shunt conductance of each port i is increased by y_{ii}'' to obtain the realisation corresponding to Y_k. It is obvious that this new network has the same modified matrix as the one realising Y_k'.

In a general case, a number of different sets of potential factors satisfying the inequalities in eqn. 29, and hence a range of different modified cut-set matrices, can be found. As pointed out in Section 3, when one of the matrices of the given set $Y_1, Y_2, \ldots, Y_p$ is not superdominant, $\epsilon_{min} = 0$, and the choice is limited to $K_1 = K_2 = \ldots = K_n = \tfrac{1}{2}$.

Next we consider the question of synthesis of 2-element-kind n-port networks. Each entry $y_{ij}(s)$ in the short-circuit admittance matrix Y of a 2-element-kind network can be expanded as follows, in a general case:

LC network

$$y_{ij}(s) = y_{ij}^{(1)}s + \frac{y_{ij}^{(2)}}{s} + \sum_{r=3}^{p} \frac{2y_{ij}^{(r)}s}{s^2 + w_r^2}$$

RL network

$$y_{ij}(s) = y_{ij}^{(1)} + \frac{y_{ij}^{(2)}}{s} + \sum_{r=3}^{p} \frac{y_{ij}^{(r)}}{s + \sigma_r}$$

RC network

$$y_{ij}(s) = y_{ij}^{(1)}s + y_{ij}^{(2)} + \sum_{r=3}^{p} \frac{y_{ij}^{(r)}s}{s + \sigma_r}$$

The nth-order real matrices $[y_{ij}^{(1)}], [y_{ij}^{(2)}], \ldots, [y_{ij}^{(p)}]$ are termed the residue matrices. When the residue matrices are dominant, they can be realised by resistive K networks having the same modified cut-set matrix by the procedure outlined. All the admittances of the network realising a particular residue matrix are then multiplied by the appropriate function of s, and the resistive network is converted into a 2-element-kind network. The parallel combination of all these networks will have a short-circuit admittance matrix equal to the given Y matrix.

If the residue matrices are not dominant, the admittance level of some or all of the transfer admittances may be first scaled down by appropriate factors so that the residue matrices are dominant. An n-port network having this modified Y matrix retains the same poles and zeros for the driving-point and transfer admittances as specified. However, the transfer admittances are realised with appropriate scale factors.

Once a network realising a given Y matrix is obtained, a range of continuously equivalent n-port networks can always be obtained. For this purpose, the general method given in Reference 3 or, preferably, the explicit formulas applicable to K networks as given in Reference 2 may be used. An example is next worked out to illustrate the synthesis procedure for 2-element-kind n-port networks.

Example 2

We consider the realisation of the following short-circuit admittance matrix Y:

$$Y = \begin{bmatrix} \dfrac{8s^4+21s^2+5}{s^3+s} & \dfrac{-3s^2-1}{s} & \dfrac{2s^4+8s^2+2}{s^3+s} \\[2ex] \dfrac{-3s^2-1}{s} & \dfrac{6s^4+21s^2+5}{s^3+s} & \dfrac{-s^4+6s^2+1}{s^3+s} \\[2ex] \dfrac{2s^4+8s^2+2}{s^3+s} & \dfrac{-s^4+6s^2+1}{s^3+s} & \dfrac{5s^4+23s^2+4}{s^3+s} \end{bmatrix}$$

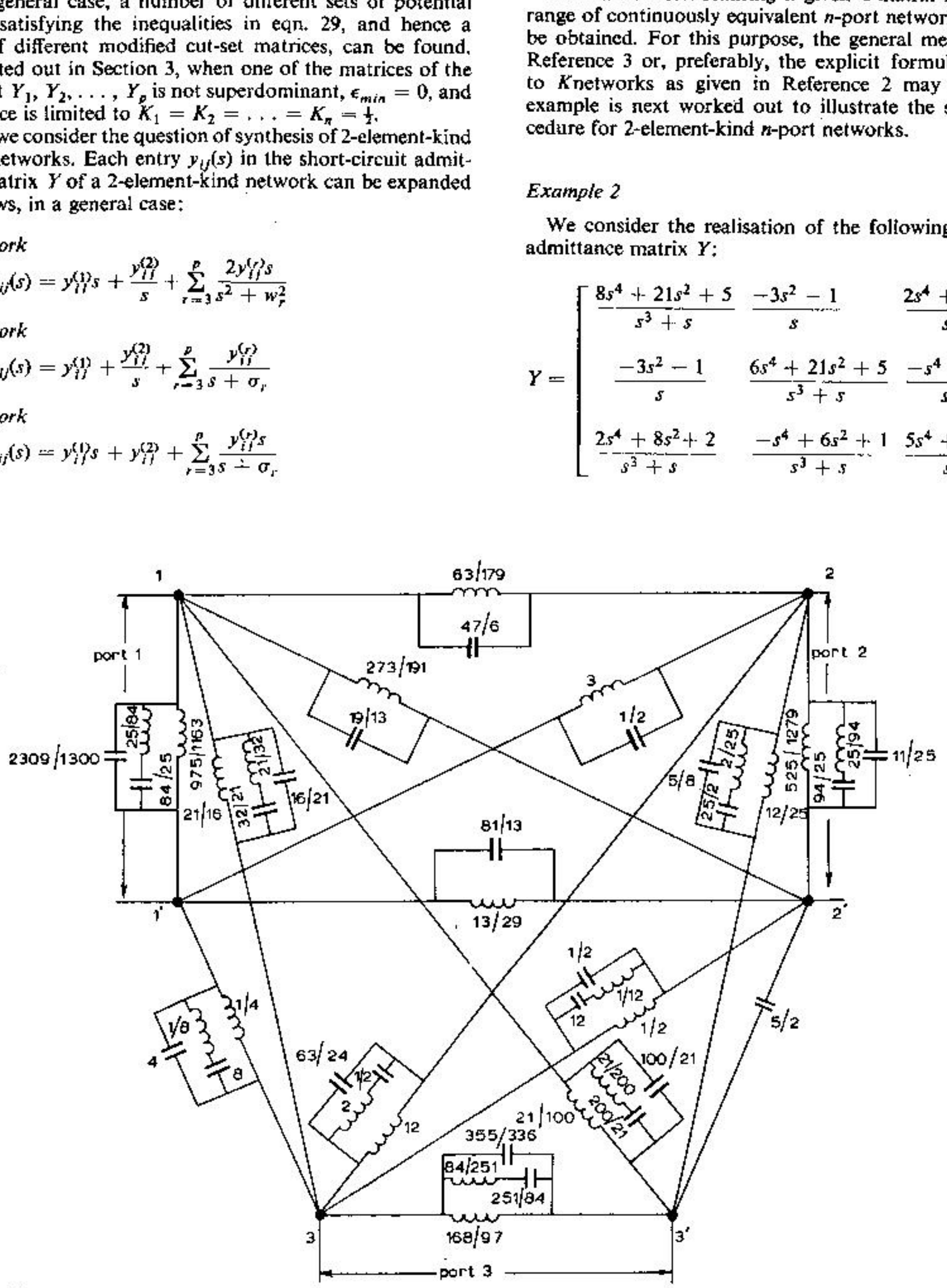

Fig. 3

3-port network realising the matrix Y of example 2

This can be split up as follows:

$$Y = s\begin{bmatrix} 8 & -3 & 2 \\ -3 & 6 & -1 \\ 2 & -1 & 5 \end{bmatrix} + \frac{1}{s}\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 1 \\ 2 & 1 & 4 \end{bmatrix} + \frac{2s}{s^2+1}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 5 & 3 \\ 2 & 3 & 7 \end{bmatrix}$$

$$= sY_1 + \frac{1}{s}Y_2 + \frac{2s}{s^2+1}Y_3$$

To make $\Delta_i \geqslant \Delta_j$ for $j > i$ for all the matrices, we split Y_2 as follows:

$$Y_2 = Y_2' + Y_2'' = \begin{bmatrix} 5 & -1 & 2 \\ -1 & 4 & 1 \\ 2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We next consider the realisation of Y_1, Y_2' and Y_3 by Knetworks having the same potential factors $\epsilon_{min} = \frac{1}{4}$, and we choose $K_1 = 0{\cdot}58$, $K_2 = 0{\cdot}52$ and $K_3 = 0{\cdot}5$. The following conductances for the resistive 3-port networks realising Y_1, Y_2 and Y_3 are then obtained:

Network 1 realising Y_1

$$a_{12} = \frac{47}{6},\ b_{12} = \frac{1}{2},\ c_{12} = \frac{19}{13},\ d_{12} = \frac{81}{13}$$

$$a_{13} = \frac{16}{21},\ b_{13} = 4,\ c_{13} = \frac{100}{21},\ d_{13} = 0$$

$$a_{23} = \frac{63}{24},\ b_{23} = \frac{1}{2},\ c_{23} = \frac{5}{8},\ d_{23} = \frac{5}{2}$$

$$g_1 = \frac{2309}{1300},\ g_2 = 0{\cdot}44,\ g_3 = \frac{355}{336}$$

Network 2 realising Y_2'

$$a_{12} = \frac{179}{63},\ b_{12} = \frac{1}{3},\ c_{12} = \frac{191}{273},\ d_{12} = \frac{29}{13}$$

$$a_{13} = \frac{16}{21},\ b_{13} = 4,\ c_{13} = \frac{100}{21},\ d_{13} = 0$$

$$a_{23} = \frac{1}{12},\ b_{23} = 2,\ c_{23} = \frac{50}{24},\ d_{23} = 0$$

$$g_1 = \frac{1163}{975},\ g_2 = \frac{754}{525},\ g_3 = \frac{291}{504}$$

Network 3 realising Y_3

$$a_{12} = b_{12} = c_{12} = d_{12} = 0$$

$$a_{13} = \frac{16}{21},\ b_{13} = 4,\ c_{13} = \frac{100}{21},\ d_{13} = 0$$

$$a_{23} = \frac{1}{4},\ b_{23} = 6,\ c_{23} = \frac{25}{4},\ d_{23} = 0$$

$$g_1 = \frac{42}{25},\ g_2 = \frac{47}{25},\ g_3 = \frac{251}{168}$$

The network realising $Y_2 = Y_2' + Y_2''$ can be obtained from network 2 by connecting an additional conductance of 1 across port 2 of network 2. The networks realising sY_1, and Y_2/s and $2s/(s^2+1)/Y_3$ can be obtained by converting the conductances of the networks realising Y_1, Y_2 and Y_3 into the appropriate admittance functions. The overall 3-port network realising the short-circuit admittance matrix Y is shown in Fig. 3.

6 Conclusions

While the techniques used in the synthesis of resistive networks, as illustrated by example 1, are useful from the point of view of their extension to general RLCnetworks, the synthesis of resistive networks is in itself important in electrical analogue representations of nonelectrical systems, e.g. mechanical systems.

The method of synthesis of a Ymatrix of an RLCnetwork as illustrated by example 2 would be useful in situations where the voltage-ratio transfer functions between some specified pairs of ports are specified, e.g. in filter networks. In such cases, a suitable Ymatrix may be chosen to satisfy these specifications and to permit the realisation of the network by the method used in example 2. Furthermore, a 3-port network, if terminated by certain prescribed active or passive elements at one or two of its ports, reduces to a 2-port or 1-port network. If the 1-port or 2-port network specifications require that the 3-port network shall consist of only a certain class of passive elements, e.g. LCelements only, the given specifications may be used to form a suitable Ymatrix for the 3-port network, and the 3-port network may be realised.

The usefulness of the concept of the modified cut-set matrix in the problem of synthesis of RLC n-port networks is thus demonstrated. The method given to realise a real symmetric dominant matrix by a Knetwork and its extension to the realisation of 2-element-kind n-port networks completes the generalisation of the realisation procedure originally given by Foster.

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9 Appendix

Case A

Let the y parameters and potential factors satisfy the following relationships:

$$y_{11} \geqslant y_{22};\ K_1 \geqslant K_2 \geqslant \tfrac{1}{2} \text{ and } (y_{22})(1-K_1) \geqslant K_2|y_{12}|$$

Then the following inequalities hold:

$y_{12} < 0$

$$\frac{y_{12}}{1-K_2} \leqslant \frac{y_{12}}{K_1} \leqslant \frac{y_{12}(K_1-K_2)}{K_1(1-K_2)} \leqslant 0 \leqslant \frac{y_{22}(1-K_1)}{(1-K_2)} + y_{12} \leqslant \frac{y_{11}K_2}{K_1} + y_{12}$$

$y_{12} > 0$

$$0 \leqslant \frac{y_{12}(K_1-K_2)}{K_1(1-K_2)} \leqslant \frac{y_{12}}{K_1} \leqslant \frac{y_{12}}{1-K_2} \leqslant \frac{y_{22}(1-K_1)}{1-K_2} + y_{12} \leqslant \frac{y_{11}K_2}{K_1} + y_{12}$$

A nonnegative b, such that

$$\frac{y_{12}}{1-K_2} \leqslant b \leqslant \frac{y_{22}(1-K_1)}{1-K_1} + y_{12}$$

satisfies the constraints in eqn. 11.

Case B

Let the y parameters and potential factors satisfy the following relationships:

$$y_{22} \geqslant y_{11};\ \tfrac{1}{2} \geqslant K_1 \geqslant K_2 \text{ and } y_{11}K_2 \geqslant (1-K_1)|y_{12}|$$

Then the following inequalities hold:

$y_{12} < 0$

$$\frac{y_{12}}{K_1} \leqslant \frac{y_{12}}{1-K_2} \leqslant \frac{y_{12}(K_1-K_2)}{(1-K_2)K_1} \leqslant 0 \leqslant y_{11}\frac{K_2}{K_1} + y_{12} \leqslant \frac{y_{22}(1-K_1)}{1-K_2} + y_{12}$$

$y_{12} > 0$

$$0 \leqslant \frac{y_{12}(K_1-K_2)}{(1-K_2)K_1} \leqslant \frac{y_{12}}{1-K_2} \leqslant \frac{y_{12}}{K_1} \leqslant y_{11}\frac{K_2}{K_1} + y_{12} \leqslant \frac{y_{22}(1-K_1)}{1-K_2} + y_{12}$$

A nonnegative b such that

$$\frac{y_{12}}{K_1} \leqslant b \leqslant y_{11}\frac{K_2}{K_1} + y_{12}$$

satisfies eqn. 11.