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Analysis and Synthesis of the K- and Y-Matrices of Resistive n-Port Networks

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With 3 Figures (Received 25th May 1973)

I. Introduction

The potential factor matrix K of an n-port network was first introduced in connection with establishing a criterion for the proper parallel connection of n-port networks [1]. An extensive use of the concept of potential factors was later made in the realisation of a real symmetric dominant matrix as the Y-matrix of an n-port network [2, 3]. Certain aspects of the relationship between the modified cutset matrix and the potential factors of an n-port network were dealt with in [4]. Recently *Lempel* and *Cederbaum* have discussed the synthesis of K-matrices [2] of resistive n-port networks [5]. In a more recent paper [6], the usefulness of the concept of potential factors in the realisation of Y-matrices of n-port networks and the synthesis of K-matrices of $(n + 2)$-node resistive n-port networks have been discussed.

In this paper analysis and synthesis of K- and Y-matrices of resistive n-port networks are considered. In Section II, an equation relating the modified cutset matrix and the K-matrix of an n-port network and certain results regarding port-vertex equivalent n-port networks are given. A procedure is given in Section III for the generation of padding n-port networks. Synthesis of K and Y matrices is discussed in Section IV. A lower bound on the number of conductances required for the realisation of Y-matrices of n-port networks having a prescribed port configuration is also obtained in Section IV.

Unless stated otherwise we follow the notation used in [6].

II. Relationship between the modified cutset Matrix and the K-Matrix of an n-Port Network

We consider a resistive n-port network N having a port configuration T. We assume, without loss of generality, that N contains no internal vertices. The linear graph of N will be denoted by G. Let the port configuration T be in p parts T_i, $i = 1, 2, \ldots, p$. Let T_0 be a tree of G and let T be a subgraph of T_0. The edges of T will be referred to as port branches and the remaining edges of T_0 will be called non-port branches (n. p. b.). The j^{th} port of N and the corresponding port branch will be both denoted by p_j. The set of branches of T_i and the corresponding set of ports will be both denoted by P_i.

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[2] In the definition used by *Lempel* and *Cederbaum*, all the diagonal entries of the K-matrix are equal to zero.

Let C_0 be the fundamental cutset matrix of G with respect to T_0. Let $C_1(C_2)$ be the submatrix of C_0 such that the rows of $C_1(C_2)$ correspond to port (non-port) branches. If V_e, V_p, and V_n denote the column matrices of edge voltages, port voltages and non-port branch voltages, then

$$\begin{aligned} V_n &= M^t V_p \\ &= [m_{ij}]^t\, V_p \end{aligned} \tag{1}$$

where m_{ij} is the voltage across the j^{th} non-port branch when port i is excited with a source of unit voltage and all the other ports are short-circuited, and

$$V_e = C^t V_p \tag{2}$$

where C is the modified cutset matrix of N and is given by [7]

$$C = C_1 + MC_2. \tag{3}$$

We now proceed to obtain an equation relating the matrix M to the potential factor matrix K.

We first define an $(n \times p)$ matrix

$$\overline{K} = [\overline{K}_{ij}] = \begin{bmatrix} \overline{K}_1 \\ \overline{K}_2 \\ \vdots \\ \overline{K}_p \end{bmatrix} \tag{4}$$

as follows:

a) i^{th} row of $\overline{K}$ corresponds to port p_i;

b) i^{th} column of $\overline{K}$ corresponds to the set of ports P_i;

c) the rows of the submatrix $\overline{K}_i$ corresponds to the ports of P_i;

d) if $j \neq i$, then the j^{th} column of $\overline{K}_i$ is equal to some column of K_{ij};

e) if $j = i$, then the j^{th} column of $\overline{K}_i$ consists of 1's only.

From the above definition of $\overline{K}$, we observe that if $p_i \notin P_j$. then $\overline{K}_{ij}$ represents the voltage of P_j with respect to the negative reference terminal of port p_i, when p_i is excited with a source of unit voltage and all the other ports are short-circuited. Also, the port configuration T and $\overline{K}$ completely specify K.

Let $\overline{T}$ be the linear graph obtained after short-circuiting all the port branches of T_0. $\overline{T}$ will have p vertices, v_i, $i = 1, 2, \ldots, p$, the vertex v_i corresponding to the set of ports P_i. The $(p-1)$ non-port branches of T_0 will form the edges of $\overline{T}$. Let A be the incidence matrix of $\overline{T}$, the i^{th} row of A corresponding to v_i and the j^{th} column corresponding to the j^{th} non-port branch.

Let $\overline{T}_i$ be the graph obtained from T_0 after short-circuiting all the port branches except p_i. If $p_i \in P_j$, then the $(p-1)$ vertices v_i, $i = 1, 2, \ldots, p$, $i \neq j$ and the two vertices of p_i will constitute the vertex set of $\overline{T}_i$.

Following are the possible ways in which the r^{th} non-port branch (r^{th} n. p. b.) can be situated in $\overline{T}_i$, with respect to p_i.

a) r^{th} n. p. b. is not incident at the vertices of p_i (Fig. 1a).

b) r^{th} n. p. b. is incident at and oriented towards the positive reference terminal of p_i (Fig. 1b).

c) r^{th} n. p. b. is incident at and oriented away from the positive reference terminal of p_i (Fig. 1c).

d) r^{th} n. p. b. is incident at and oriented towards the negative reference terminal of p_i (Fig. 1d).

e) r^{th} n. p. b. is incident at and oriented away from the negative reference terminal of p_i (Fig. 1e).

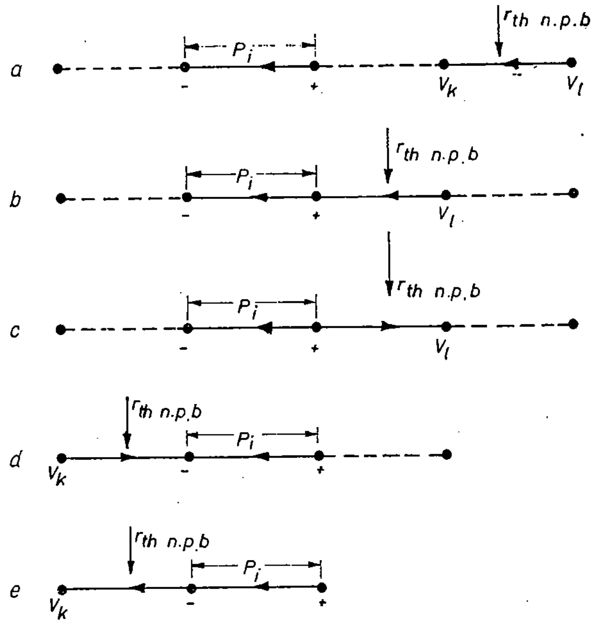

Fig. 1

We next define an $(n \times p - 1)$ matrix $\bar{A} = [\bar{a}_{ij}]$ as follows:

a) i^{th} row of $\bar{A}$ corresponds to p_i and the j^{th} column corresponds to the j^{th} n. p. b.

b) $\bar{a}_{ir} = 0$, if in $\bar{T}_i$, either (i) the r^{th} n. p. b. is not incident at the vertices of p_i, or (ii) r^{th} n. p. b. is incident at the positive reference terminal of p_i.

c) $\bar{a}_{ir} = 1$, if in $\bar{T}_i$, r^{th} n. p. b. is incident at and oriented towards the negative reference terminal of p_i.

d) $\bar{a}_{ir} = -1$, if in $\bar{T}_i$, r^{th} n. p. b. is incident at and oriented away from the negative reference terminal of p_i.

We then have the following theorem.

Theorem 1:

$$M = \bar{K}A + \bar{A}.$$

Proof:

The (i, r) entry m_{ir} of M represents, by definition, the voltage across the r^{th} n. p. b. when port p_i is excited with a source of unit voltage and all the other ports are short-circuited. We shall denote the (i, r) entry of $\bar{K}A$ as $(\bar{K}A)_{i,r}$. We shall consider the five possible ways enumerated earlier, in which the r^{th} n. p. b. can be situated in $\bar{T}_i$ (Figs. 1 (a), (b), (c), (d) and (e)) and obtain in each case m_{ir} $(\bar{K}A)_{i,r}$ and $\bar{a}_{ir}$.

Case A: In $\bar{T}_i$ rth n. p. b. is situated with respect to p_i as in Fig. 1 (a).

$$m_{ir} = \bar{k}_{il} - \bar{k}_{ik}$$
$$(\bar{K}A)_{i,r} = \bar{k}_{il} - \bar{k}_{ik}$$
$$\bar{a}_{ir} = 0.$$

Case B: In $\overline{T}_i$ r^{th} n. p. b. is situated with respect to p_i as in Fig. 1 (b).

$$m_{ir} = \bar{k}_{il} - 1$$
$$(\overline{K}A)_{i,r} = \bar{k}_{il} - 1$$
$$\bar{a}_{ir} = 0$$

Case C: In $\overline{T}_i$ r^{th} n. p. b. is situated with respect to p_i as in Fig. 1 (c).

$$m_{ir} = 1 - \bar{k}_{il}$$
$$(\overline{K}A)_{i,r} = 1 - \bar{k}_{il}$$
$$\bar{a}_{ir} = 0.$$

Case D: In $\overline{T}_i$ r^{th} n. p. b. is situated with respect to p_i as in Fig. 1 (d).

$$m_{ir} = \bar{k}_{ik}$$
$$(\overline{K}A)_{i,r} = \bar{k}_{ik} - 1$$
$$\bar{a}_{ir} = 1.$$

Case E: In $\overline{T}_i$ r^{th} n. p. b. is situated with respect to p_i as in Fig. 1 (e).

$$m_{ir} = -\bar{k}_{ik}$$
$$(\overline{K}A)_{i,r} = 1 - \bar{k}_{ik}$$
$$\bar{a}_{ir} = -1.$$

We observe that in all the cases considered above

$$m_{ir} = (\overline{K}A)_{i,r} + \bar{a}_{ir}.$$

Hence the theorem.

It follows from theorem (1) and equation (3) that

$$C = C_1 + (\overline{K}A + \overline{A})\, C_2. \tag{5}$$

We note that $\overline{K} = K$, in the case of $2n$-node n-port networks. Hence in that case

$$C = C_1 + (KA + \overline{A})\, C_2. \tag{6}$$

Eq. (5) and (6) are respectively similar to Eq. (61) and (10) of reference [5]. The latter equations involve the use of certain submatrices of the matrix relating the incidence and fundamental cutset matrices of a graph obtained from G.

Consider, next, an n-port network, N^* constructed on N. Let T^*, the port configuration of N^*, also be in p parts T_i^*, $i = 1, \ldots, p$, such that the vertices of T_i^* are the same as those of T_i. The n-port networks N and N^* defined as above will be referred to as port-vertex equivalent n-port networks.

Let C^* be the modified cutset matrix of N^*. Let Y^*, V_{p}^* and M^* be defined similarly. If

$$V_{\text{p}} = A^t V_{\text{p}}^* \tag{7}$$

then it is easy to show that

$$M^* = AM \tag{8}$$
$$C^* = AC \tag{9}$$

and

$$Y^* = AYA^t. \tag{10}$$

Further, if N is a padding n-port network, then it follows from (10) that N^* is also a padding n-port network.

III. Synthesis of padding n-Port Networks

We obtain, in this section, a procedure for the generation of padding n-port networks, having specified potential factors and a prescribed port configuration.

We shall assume, without loss of generality, that each connected part T_i, $i = 1, 2, \ldots, p$ of the port configuration of the required padding n-port network N is a lagrangian tree. The set of vertices of T_i will be denoted as $i_0, i_1, i_2, \ldots, i_{n_i}$, with i_0 as the star vertex of T_i. The m^{th} port of P_i will be denoted by $P_{i(m)}$. i_m and i_0 are the positive and negative reference terminals of $P_{i(m)}$. T_i and the polarities of $P_{i(m)}$ are as shown in Fig. 2.

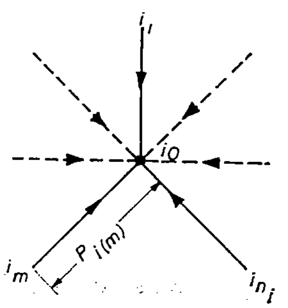

Fig. 2

The conductance of the edge connecting the vertices i_k and j_m of N will be denoted by g_{ikjm}. We further define S_{ikj} and S_{ij} as follows:

$$S_{ikj} = \sum_{m=0}^{n_j} g_{ikjm}, \; j \neq i$$

and

$$S_{ij} = \sum_{k=0}^{n_i} S_{ikj}, \; j \neq i$$
$$= \sum_{m=0}^{n_j} S_{jmi}.$$

The network obtained from N after short circuiting all the ports will be denoted by $\bar{N}$, and the network obtained after short-circuiting all the ports except $P_{i(m)}$ and connecting a source of unit voltage across $P_{i(m)}$ will be denoted by $N_{i(m)}$. We observe that (i) the p vertices v_i, $i = 1, 2, \ldots, p$ will constitute the vertex set of $\bar{N}$. (ii) the $(p-1)$ vertices v_r, $r = 1, 2, \ldots, p$, $r \neq i$ and the vertices of $P_{i(m)}$ will constitute the vertex set of $N_{i(m)}$ and (iii) C_2 is the fundamental cutset matrix of $\bar{N}$ with respect to $\bar{T}$. If A_i is the reduced incidence matrix of $\bar{N}$ with v_i as the reference vertex, then C_2 can be expressed as

$$C_2 = R_i A_i. \tag{11}$$

It is well known that R_i is non-singular. Further we denote by $k_{i(k),j}$, $j \neq i$, the voltage of the set of ports P_j when $P_{i(k)}$ is excited with a source of unit voltage and all the other ports are short circuited. We note that $k_{i(k),j}$ is equal to some element of the k^{th} row of K_{ij}, $j \neq i$.

Given the port configuration T and the potential factors, the modified cutset matrix C of the required padding n-port network N can be easily constructed. It has been shown that a real diagonal matrix G will represent the edge conductance matrix of a padding n-port network N if and only if the following equations are satisfied [4].

$$CGC_2^t = 0 \tag{12a}$$

$$CGC_1^t = 0 \tag{12b}$$

and

$$\det (C_2 G C_2^t) \neq 0. \tag{12c}$$

We next proceed to solve Eq. (12) for G.

Consider first Eq. (12a). This equation can be written as a set of linear equations with $S_{i_k j}$'s as unknowns. If $C_i^{(k)}$ denotes the row of C corresponding to $P_{i(k)}$, then equation (12a) can be written as

$$C_i^{(k)} G C_2^t = 0, \qquad \begin{array}{l} i = 1, 2, \ldots, p, \\ k = 1, 2, \ldots, n_i. \end{array} \tag{13}$$

From (11) and (13) we get

$$C_i^{(k)} G A_i^t = 0, \qquad \begin{array}{l} i = 1, 2, \ldots, p, \\ k = 1, 2, \ldots, n_i. \end{array} \tag{14}$$

We note that Eq. (13) and (14) are equivalent since R_i is non-singular. The equation

$$C_i^{(k)} G A_i^t = 0 \quad \text{for some } i \text{ and some } k$$

represents the following set of $(p-1)$ equations.

$$S_{i_k j}(k_{i(k),j} - 1) + \sum_{\substack{m=0 \\ m \neq k}}^{n_i} S_{i_m j}\, k_{i(k),j} + \sum_{\substack{m=1 \\ m \neq j;\, m \neq i}}^{p} S_{j,m}(k_{i(k),j} - k_{i(k),m}) = 0 \tag{15}$$

$$j = 1, 2, \ldots, p, \; j \neq i.$$

Eq. (15) can be easily identified as Kirchhoff's current law equation for $N_{i(k)}$ at the $(p-1)$ vertices v_r, $r = 1, 2, \ldots, p$, $r \neq i$. Solving (15) for $S_{i_k j}$ and generalising the result we get

$$S_{i_k j} = S_{i,j} k_{i(k),j} + \sum_{\substack{m=1 \\ m \neq i,\, m \neq j}}^{p} S_{j,m}\,(k_{i(k),j} - k_{i(k),m}) \tag{16}$$

$$i = 1, 2, \ldots, p,$$

$$k = 1, 2, \ldots, n_i$$

and

$$j = 1, 2, \ldots, p, \; j \neq i.$$

Since

$$S_{i,j} = \sum_{k=0}^{n_i} S_{i_k j}.$$

We get from (16)

$$\begin{aligned} S_{i_0 j} &= S_{ij} - \sum_{k=1}^{n_i} S_{i_k j} \\ &= S_{ij}\left(1 - \sum_{k=1}^{n_i} k_{i(k),j}\right) - \sum_{k=1}^{n_i} \sum_{\substack{m=1 \\ m \neq i,\, m \neq j}}^{p} S_{j,m}\,(k_{i(k),j} - k_{i(k),m}) \\ & i = 1, 2, \ldots, p \quad \text{and} \quad j = 1, 2, \ldots, p,\ j \neq i. \end{aligned} \tag{17}$$

Eq. (16) and (17) can be used to evaluate all $S_{i,j}$'s after assuming arbitrary values for $S_{i,j}$'s. $S_{i_k j}$'s so obtained will satisfy Eq. (12a).

We next proceed to solve Equation (12b) and obtain expressions for the edge conductances of N in terms of $S_{i_k j}$'s.

Let $C_{1,j(m)}$ denote the row of C_1 corresponding to $P_{j(m)}$. Then, taking into account the symmetry of the short-circuit conductance matrix CGC_1^t, the following sets of equations are obtained from Eq. (12b).

$$C_i^{(k)} G C_{1,j(m)}^t = 0, \quad \begin{aligned} & i = 1, 2, \ldots, p-1, \\ & k = 1, 2, \ldots, n_i, \\ & j = 2, 3, \ldots, p,\ j > i, \\ & m = 1, 2, \ldots, n_j. \end{aligned} \tag{18a}$$

$$C_i^{(k)} G C_{1,i(m)}^t = 0, \quad \begin{aligned} & i = 1, 2, \ldots, p, \\ & k = 1, 2, \ldots, n_i - 1, \\ & m = 2, 3, \ldots, n_i,\ m > k \end{aligned} \tag{18b}$$

and

$$C_i^{(k)} G C_{1,i(k)}^t = 0, \quad \begin{aligned} & i = 1, 2, \ldots, p, \\ & k = 1, 2, \ldots, n_i. \end{aligned} \tag{18c}$$

Consider the equation

$$C_i^{(k)} G C_{1,j(m)}^t = 0, \quad \text{for some } i, k, j > i, \text{ and } m.$$

This equation can be written as

$$g_{i_k j_m}(k_{i(k),j} - 1) + \sum_{\substack{r=0 \\ r \neq k}}^{n_i} g_{i_r j_m} k_{i(k),j} + \sum_{\substack{r=1 \\ r \neq i \\ r \neq j}}^{p} S_{j_m r}(k_{i(k),j} - k_{i(k),r}) = 0. \tag{19}$$

Solving (19) for $g_{i_k j_m}$ and generalising the result, we get

$$\begin{aligned} g_{i_k j_m} &= S_{j_m k}\, k_{i(k),j} + \sum_{\substack{r=1 \\ r \neq i \\ r \neq j}}^{p} S_{j_m r}(k_{i(k),j} - k_{i(k),r}) \\ & i = 1, 2, \ldots, p-1 \\ & k = 1, 2, \ldots, n_i \\ & j = 2, 3. \ldots, p, j > i \\ & m = 1, 2, \ldots, n_j. \end{aligned} \tag{20}$$

Values for $g_{i_kj_m}$'s obtained by using (20) will satisfy (18a). Further $g_{i_0j_m}$ and $g_{i_kj_0}$ and $g_{i_0j_0}$ can be obtained as

$$g_{i_0j_m} = S_{j_mi} - \sum_{k=1}^{n_i} g_{i_mi_k}$$

$$g_{i_kj_0} = s_{i_kj} - \sum_{m=1}^{n_j} g_{i_kj_m}$$

and

$$\begin{aligned} g_{i_0j_0} &= S_{i_0j} - \sum_{m=1}^{n_j} g_{i_0j_m} \\ i &= 1, 2, \ldots, p-1 \\ k &= 1, 2, \ldots, n_i \\ j &= 2, 3, \ldots, p, \; j > i \\ m &= 1, 2, \ldots, n_j. \end{aligned} \tag{21}$$

Eq. (20) and (21) will enable us to obtain conductances of edges connecting vertices in different T_i's.

We then consider equation (18b). The equation

$$C_i^k G C_{1,i(m)}^t = 0 \quad \text{for some } i, k \text{ and } m > k$$

can be written as

$$-g_{i_ki_m} - \sum_{\substack{j=1 \\ j \neq i}}^{p} S_{i_mj} k_{i(k),j} = 0. \tag{22}$$

Solving Eq. (22) for $g_{i_ki_m}$ and generalising the result we get

$$\begin{aligned} g_{i_ki_m} &= \sum_{\substack{j=1 \\ j \neq i}}^{p} S_{i_mj} k_{i(k),j} \\ i &= 1, 2, \ldots, p, \\ k &= 1, 2, \ldots, n_i - 1, \\ m &= 2, 3, \ldots, n_i, \; m > k. \end{aligned} \tag{23}$$

Values for $g_{i_ki_m}$'s obtained using (23) will satisfy Eq. (18b).

Finally, we consider equation (18c). The equation

$$C_i^{(k)} G C_{1,i(k)}^t = 0, \quad \text{for some } i \text{ and some } k$$

can be written as

$$g_{i_ki_0} + \sum_{\substack{j=1 \\ j \neq i}}^{p} S_{i_kj}(1 - k_{i(k),j}) + \sum_{\substack{m=1 \\ m \neq k}}^{n_i} g_{i_ki_m} = 0. \tag{24}$$

From Eq. (24) we get

$$g_{i_ki_0} + \sum_{\substack{m=1 \\ m \neq i}}^{n_i} g_{i_ki_m} + \sum_{\substack{j=1 \\ j \neq i}}^{p} (k_{i(k),j} - 1) S_{i_kj}. \tag{25}$$

We get from Eq. (23) and (25)

$$\begin{aligned} g_{i_k i_0} &= \sum_{\substack{m=1\\ m\neq k}}^{n_i} \sum_{\substack{j=1\\ j\neq i}}^{p} S_{i_m j} k_{i(k),j} - \sum_{\substack{j=1\\ j\neq i}}^{p} (1 - k_{i(k),j})\, S_{i_k j} \\ &= \sum_{\substack{j=1\\ j\neq i}}^{p} k_{i(k),j} S_{i_0,j}\,. \end{aligned} \tag{26}$$

The last step in Eq. (26) follows after equating to zero the sum of the currents in those edges of $N_{i(k)}$, connecting vertices in T_i to all other vertices. Generalising the result obtained in Eq. (26) we get,

$$\begin{aligned} g_{i_k i_0} &= -\sum_{\substack{j=1\\ j\neq i}}^{p} k_{i(k),j} S_{i_0 j}\,. \\ i &= 1, 2, \ldots, p, \\ k &= 1, 2, \ldots, n_i\,. \end{aligned} \tag{27}$$

Values for $g_{i_k i_0}$'s obtained using Eq. (27) will satisfy (18c).

The discussions up to this point may be summarized as follows:

a) Assuming arbitrary values for $S_{i,j}$'s, determine $S_{i_k j}$'s using Eq. (16) and (17).

b) Use the values of $S_{i_k j}$'s so obtained, in Eq. (20), (21), (23) and (27), and determine the values for the edge conductances $g_{i_k j_m}$'s of N.

c) The values of edge conductances so obtained will satisfy Eq. (12a) and (12b).

We next turn to (12c). Need for padding network synthesis arises in the realisation of K and Y matrices by n-port networks having no negative conductances. If N is to be the padding n-port network of some n-port network containing no negative conductances then all $S_{i,j}$'s should be chosen nonnegative. Further if $\overline{N}$ is connected and contains no negative conductances (i.e., all S_{ij}'s are non-negative), then $(C_2 G C_2^t)$ will be nonsingular and (12c) will be satisfied. So, while selecting values for S_{ij}'s it must be ensured

i) all S_{ij}'s are non-negative, and

ii) some S_{ij}'s must be positive so that $\overline{N}$ is connected.

In the foregoing, expressions for edge conductances of N have been obtained, assuming that each connected part of T is a langrangian tree. This assumption, however, involves no loss of generality, as may be seen from the following.

If any arbitrary connected port configuration T^* and the corresponding potential factors are specified then, the potential factors corresponding to T, in which each connected part is a lagrangian tree, can be easily obtained (Equation 8 in the previous section). If an n-port padding network N having the port configuration T and the newly determined potential factors is generated then the n-port network N^* with the port configuration T^* will also be a padding network with its potential factors as specified. It may be noted N and N^* are port-vertex equivalent n-port networks.

If, however, the port configuration T^* alone is specified, then we should first generate N assuming values for all distinct S_{ij}'s as well as for all $k_{i(k),j}$'s. The n-port network N^* port-vertex equivalent to N and having the prescribed port configuration T^* will be the padding n-port network required.

This completes our discussions on the synthesis of padding n-port networks having any arbitrary connected port configuration and having specified potential factors. The usefulness of these results will be discussed in the next section.

Example 1:

It is required to generate a 3-port padding network having the port configuration shown in Fig. 3a. The potential factors of the required network should be as follows:

$$k_{1(1),2} = k_{12} = 0.5; \qquad k_{1(1),3} = k_{13} = 0.6$$
$$k_{2(1),1} = k_{21} = 0.4; \qquad k_{2(1),3} = k_{23} = 0.5$$
$$k_{3(1),1} = k_{31} = 0.3; \qquad k_{3(1),2} = k_{32} = 0.2.$$

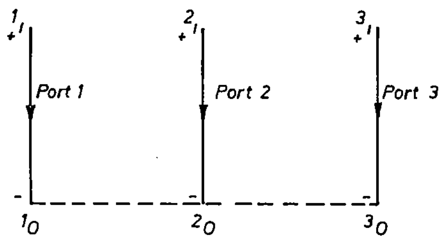

Fig. 3a

Assume $S_{i,j}$'s as follows: $S_{12} = 10$, $S_{13} = 20$; $S_{23} = 10$. Using Eq. (16) and (17) S_{i_kj}'s are obtained as

$$S_{1_12} = 4; \; S_{1_02} = 6; \; S_{1_13} = 13; \; S_{1_03} = 7;$$
$$S_{2_11} = 2; \; S_{2_01} = 8; \; S_{2_13} = 7; \; S_{2_03} = 3;$$
$$S_{3_11} = 7; \; S_{3_01} = 13; \; S_{3_12} = 1; \; S_{3_02} = 9.$$

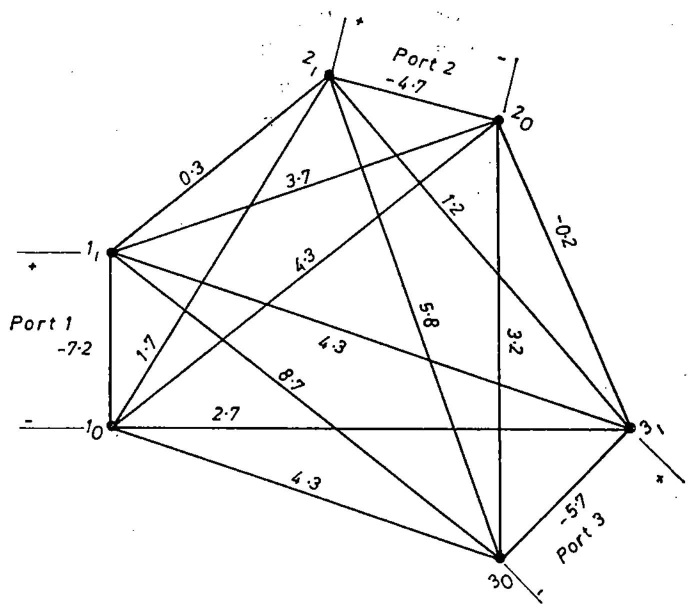

Fig. 3b

For example,

$$S_{3_12} = S_{23}k_{32} + S_{21}(k_{32} - k_{31})$$
$$= 10 \cdot 0.2 + 10 \cdot (-0.1) = 1$$

and

$$S_{3_02} = S_{23} - S_{3_12} = 9.$$

Using the values of $S_{i_k j}$'s so obtained in Eq. (20), (21), (23) and (27) we can get the edge conductances of the required 3-port network. For example

$$\begin{aligned} g_{1_1 2_1} &= S_{2_1 1} k_{12} + S_{2_1 3}(k_{12} - k_{13}) \\ &= 2 \cdot 0.5 - 7 \cdot 0.1 = 0.3 \\ g_{1_1 1_0} &= -S_{1_0 2} k_{12} - S_{1_0 3} k_{13} \\ &= -6 \cdot 0.5 - 7 \cdot 0.6 = -7.2. \end{aligned}$$

The required 3-port padding network is shown in Fig. 3b.

IV. Synthesis of K- and Y-Matrices of n-Port Networks

a) Synthesis of the K-Matrix

Synthesis of the potential factor matrix K of an n-port network requires the solution of the following two problems:

i) Determination of the port configuration T appropriate to K.

ii) Determination of an n-port network containing no negative conductances and having the port configuration T and the specified K-matrix.

In their solution of the first problem, *Lempel* and *Cederbaum* [5] first assume the port configuration to be in n parts and then obtain the modified cutset matrix, appropriate to the assumed port configuration and the given K-matrix. This modified cutset matrix can be determined either by using Eq. (10) of reference [5] or Eq. (6) of this paper or by inspection of the assumed port configuration and the given K matrix. From the modified cutset matrix so obtained, the port configuration T is determined by the application of a simple procedure, which yields a unique port configuration T for a given K matrix.

To solve the second problem, *Lempel* and *Cederbaum* first determine the modified cutset matrix C, appropriate to the port configuration T and the specified K matrix. Then linear programming technique is applied to obtain a non-negative G, if one exists, satisfying the equation

$$CGC_2^t = 0.$$

In this section, we give a new necessary and sufficient condition to test the existence of a resistive n-port network containing no negative conductances and having a specified K-matrix and a port configuration T appropriate to the matrix K. We assume, without loss of generality, that each connected part of the port configuration T is a lagrangian tree.

Let the column matrices of $(S_{i_k j})_p$'s and $(S_{ij})_p$'s of an n-port network N_p be denoted by $\overline{S}_p$ and S_p respectively. Eq. (16) and (17) can be together written in matrix form as

$$\overline{S}_p = P S_p$$

where each entry of the matrix P is a linear combination of some potential factors. If $(S_{ij})_p$'s are such that N_p is connected then the corresponding S_p will be called non-trivial. It is shown in [6] that

i) if all $(S_{i_k j})_p$'s of a padding n-port network N_p are non-negative then a network of departure N_d can be found so that the parallel combination N of N_p and N_d contains no negative conductances;

ii) an n-port network and its padding network have the same K-matrix, and

iii) $(S_{i_k j})_p = S_{i_k j}$ and $(S_{ij})_p = S_{ij}$.

Theorem (2) then follows.

Theorem 2

Let each connected part of the port configuration T appropriate to a given potential factor matrix K be a lagrangian tree. The matrix K can be realised by a resistive n-port network containing no negative conductances if and only if there exists a non-trivial S such that $S \geqq 0$ and $PS \geqq 0$.

Following steps may then be used for the synthesis of a K-matrix:

i) Obtain a non-trivial value of S, if it exists, equal to S_a such that $S_a \geqq 0$ and $PS_a \geqq 0$ [9, 14].

ii) Construct a padding n-port network N_p having the matrix K as its potential factor matrix and such that its $\overline{S}$ matrix is equal to PS_a.

iii) Determine a suitable N_d so that the parallel combination N of N_d and N_p contains no negative conductances.

iv) The network N realizes the matrix K.

We now wish to draw attention to the following.

1. According to the procedure given in [6] to determine a suitable N_d for a given N_p in which all $(S_{i_k j})_p$'s are non-negative

$$(g_{i_k i_m})_d = -(g_{i_k i_m})_p \tag{28}$$

and

$$(g_{i_k j_m})_d = -(g_{i_k j_m})_p + \frac{S_{i_k j} S_{j_m i}}{S_{ij}}, \quad i \neq j. \tag{29}$$

Since

$$g_{i_k j_m} = (g_{i_k j_m})_d + (g_{i_k j_m})_p \quad \text{for all } i \text{ and } j,$$

we get

$$g_{i_k i_m} = 0 \tag{30}$$

and

$$g_{i_k j_m} = \frac{S_{i_k j} S_{j_m i}}{S_{ij}} \text{ for all } i \text{ and } j,\ j \neq i. \tag{31}$$

Thus determination of N requires the evaluation of only $(g_{i_k j_m})$'s using (31).

2. It can be shown using Eq. (16) that for every vertex i_k there exists a j such that $S_{i_k j}$ is non-negative if all S_{ij}'s are non-negative. Thus of the $(n + p) \cdot (p - 1)$ elements of the vector $\overline{S}$, $(n + p)$ elements will be non-negative if all S_{ij}'s are non-negative. Hence the total number of constraints involved in the solution of the linear program implied in theorem (2) is only $(n + p)\,(p - 2)$. In contrast the number of constraints used in the procedure given in [5] is $n\,(p - 1)$. It may be noted that for

$$n > p\,(p - 2)$$

the procedure given in this section for K-matrix synthesis involves a smaller number of constraints than used by *Lempel* and *Cederbaum* [5]. Further the present procedure involves $\frac{p(p-1)}{2}$ number of unknowns which is less than the minimum number of unknowns, namely $2p(p - 1)$ used in [5].

The new approach given in this section for K-matrix synthesis provides a greater insight into the nature of the K-matrix synthesis problem. In fact following the same approach a simple necessary and sufficient condition has already been obtained for the synthesis of the K-matrices of $(n + 2)$-node n-port networks. Further, since this procedure essentially requires the synthesis of a suitable padding network having a specified K-matrix, it can be readily used in Y-matrix synthesis as discussed in Section IV (b).

b) Synthesis of the Y-Matrices of n-port Networks with more than (n + 1) Nodes

The only approach available for the synthesis of the Y-matrices of n-port resistive networks having more than $(n + 1)$-nodes is due to *Guillemin* [8]. This approach essentially requires the determination of a suitable padding n-port network N_p for a given network of departure N_d. As a result a number of procedures have been proposed in the past for the generation of padding n-port networks [8, 10, 11, 12]. The procedure for padding network generation given in Section III is yet another contribution in this direction.

A significant feature of this new procedure is that all the parameters used herein namely S_{ij}'s and $k_{i(k),j}$'s can be readily identified with certain quantities of the padding network N_p to be realized. Also these quantities happen to be the same for both N_p and N. The procedures for padding network synthesis given in [8, 10, 11], and [12] do not permit such straightforward identification for all the parameters. This feature of the new procedure is of help in the synthesis of Y matrices of RLC n-port networks. In the synthesis of such networks, it is required to realise a set of real symmetric Y matrices by resistive n-port networks, all having the same modified cutset matrix [3], [8]. If a network N_1 realising one of these matrices is known, then all networks realising the other matrices should have the same modified cutset matrix as N_1. This leads us to the problem of synthesis of a resistive n-port network having a prescribed Y matrix, a prescribed port configuration and specified potential factors $k_{i(k),j}$'s. To solve this, we may proceed as follows. We assume that each connected part of the port configuration T is a lagrangian tree.

Let $\{g\}_d$ be the column matrix of edge conductances of the network of departure N_d with respect to the given Y and T. Let $\{g\}_p$ be the column matrix of edge conductances of a required padding network N_p. It follows from Eq. (16), (17), (20), (21), (23) and (27) that $\{g\}_p$ can be related to S, the column matrix of S_{ij}'s, as

$$\{g\}_p = QS$$

where each entry of Q is a function of $k_{i(k),j}$'s. Hence Q can be determined from the values specified for $k_{i(k),j}$'s. Since the parallel combination of N_p and N_d should contain no negative conductances, it is required that

$$\{g\}_p = QS \geqq -\{g\}_d. \tag{32}$$

If a non-negative and non-trivial value of S equal to S_a satisfying (32) exists then the column matrix $\{g\}$ of conductances of the required n-port network will be given by

$$\{g\} = \{g\}_d + QS_a.$$

Thus it follows from (32) that when T, Y and $k_{i(k),j}$'s are specified, n-port synthesis problem simplifies to one of solving a linear program. This is in contrast to the non-linear equations involved when Y and T alone are specified. Following the approach outlined above, a simple necessary and sufficient condition has already been established for the synthesis of $(n + 2)$-node n-port networks having prescribed Y and K-matrices [13]. The procedure for padding network synthesis given in [8, 10, 11, 12] will not be of help in the synthesis of Y-matrices of RLC n-port networks.

c) Lower Bound on the Number of Conductances required for the Synthesis of a Y-Matrix

We, next, establish a lower bound on the number of conductances required for the realisation of a real symmetric matrix Y as the shortcircuit conductance matrix of a resistive n-port network containing no negative conductances and having a prescribed port configuration T.

Consider, a resistive n-port network N containing no negative conductances. Let each connected part of the port configuration T of N be a lagrangian tree.

For every port $P_{i(k)}$ of N there exists a j such that

$$k_{i(k),j} \geqq k_{i(k),r} \; r = 1, 2, \ldots, p, \quad r \neq i, \; r \neq j. \tag{33}$$

Since N contains no negative conductances all $S_{i_k j}$'s are non-genative. Then it follows from Eq. (33) and (20) that for every vertex i_k, $k \neq 0$ there exists a j such that

$$(g_{i_k j_m})_p \geqq 0 \quad \text{for all} \;\; m = 0, 1, 2, \ldots, n_j. \tag{34}$$

Let N^* be an n-port network port-vertex equivalent to N. Let the star vertex of each T_i^* be different from that of T_i. Then following the same line of argument as above, we can show that for every vertex i_0 there exists a j such that

$$(g_{i_0 j_m})_p \geqq 0 \quad \text{for all} \;\; m = 0, 1, 2, \ldots, n_j. \tag{35}$$

Since the padding networks of port-vertex equivalent n-port networks are identical, we conclude from (34) and (35) that for every vertex i_k there exists a j such that

$$(g_{i_k j_m})_p \geqq 0 \quad \text{for all} \;\; m = 0, 1, 2, \ldots, n_j. \tag{36}$$

Further, the above result is valid irrespective of the port configuration.

It follows from (36) that in the case of $(n + 2)$-node n-port networks in which $p = 2$, all the conductances connecting vertices in T_1 to vertices in T_2 are non-negative. This result has already been established in [6].

Let N_d be the network of departure with respect to a given Y matrix and a port configuration T. Let $x_{i(k),j}$ be the total number of positive conductances in N_d connecting vertex i_k to all the vertices in T_j. Let

$$x_{i(k)} = \operatorname{Min}\{x_{i(k),j}, \; j = 1, 2, \ldots, p, \; j \neq i\}.$$

Theorem (3) then follows from (36).

Theorem 3

The number of conductances required for the realisation of a real symmetrix matrix Y as the short-circuit conductance matrix of a resistive n-port network having no negative conductances and having a prescribed port configuration T cannot be less than

$$1/2 \sum_{i=1}^{p} \sum_{k=0}^{n_i} x_{i(k)}.$$

V. Conclusion

The only approach available for the synthesis of resistive n-port networks having more than $(n + 1)$-nodes is due to *Guillemin* [8]. *Guillemin*'s approach essentially requires the determination of a suitable padding n-port network N_p for a given network of departure N_d so that the parallel combination of N_d and N_p contains no negative conductances. Hence a number of procedures were suggested for generation of padding n-port networks [10, 11, 12]. All these procedures express conductances of a padding network in terms of certain arbitrary parameters. It was pointed out recently that a padding n-port network can be identified as the padding network of some resistive n-port network containing no negative conductances if and only if all $S_{i_k j}$'s are non-negative [6]. In view of this it is enough if we confine our search for a suitable padding n-port network to a restricted class of these networks. Further the potential factors and S_{ij}'s of a network and its padding network are identical. So, it seemed desirable and useful to develop a procedure for generating padding n-port networks in terms of these parameters. A step in this direction was taken in [6]. In Section III of this paper the approach presented in [6] is investigated and formulas for the conductances of a padding network in terms of potential factors and S_{ij}'s are obtained. Since these formulas are in terms of potential factors it is neccessary that we know the necessary and sufficient conditions which the potential factors of a resistive n-port network having no negative conductances should satisfy. This necessity explains the interest in the analysis and the synthesis of the K-matrix of n-port networks.

In Section II, an equation relating the modified cutset matrix and the K-matrix is established. This relationship, as pointed out in Section II, is useful in view of the method used in [5] for determining the port structure pertinent to a given K-matrix.

The procedures for K-matrix synthesis given in [5] as well as in Section IV (a), of this paper require the solution of a linear program. However, the present approach involves a smaller number of unknowns and further the number of constraints used is also smaller for all $n > p\ (p - 2)$. Added to these is ist usefulness in the Y-matrix synthesis problem.

Though the Y-matrix synthesis problem looks formidable, simultaneous realisation of K and Y matrices is straightforward as shown in Section IV (b). It is shown in [15] that the problem of synthesis of a hybrid matrix reduces to one of realising an n-port network having prescribed K and Y matrices and prescribed S_{ij}'s. Thus hybrid matrix synthesis can be achieved by a straightforward application of the results of this paper.

The lower bound on the number of conductances required for the realisation of a matrix Y might help in throwing some light on *Biorci*'s conjecture.

Abstract

In this paper a new matrix equation relating the modified cutset matrix C and the potential factor matrix K, of an n-port network is first established. A new procedure for the synthesis of padding n-port networks is then given. Based on these results, a new necessary and sufficient condition is then established for the synthesis of the K-matrices of n-port networks. Application of these results in the synthesis of Y-matrices is discussed. A lower bound on the number of conductances required for Y-matrix synthesis is also given.

Zusammenfassung

In dem Artikel wird eine neue Matrizengleichung, die die modifizierte Schnittmengenmatrix C und die Knotenspannungsmatrix K eines n-Tor-Netzwerkes miteinander verknüpft, vorgestellt. Ein neuer Algorithmus für die Synthese größerer n-Tor-Netzwerke wird gegeben. Mit diesen Ergebnissen wird eine neue hinreichende und notwendige Bedingung für die Synthese der K-Matrizen von n-Tor-Netzwerken abgeleitet. Die Ergebnisse werden bei der Synthese von Y-Matrizen angewendet und diskutiert.

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