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Structure of the Reachability Problem for (0,1)-Capacitated Marked Graphs

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Abstract —**The structure of the reachability problem for (0,1)-capacitated marked graphs is studied. A purely graph-theoretic characterization of this problem is presented.**

I. Introduction

Many classical results in graph theory are equivalent to the maximum-flow minimum-cut theorem of network-flow theory [1]. These results include Tutte's characterization of maximum matchings in general graphs, Hall's theorem on bipartite matching, and Menger's theorem on connectivity. Equivalence among these problems is established by constructing appropriate (0,1)-communication networks which permit flows of values of only zero or one on each of its edges. This equivalence has made possible the design of efficient algorithms for matching and for connectivity analysis because efficient algorithms are available for computing maximum flows in (0,1)-communication networks. These pioneering works have provided the motivation for the study presented in this paper.

We study the structure of the reachability problem for (0,1)-capacitated marked graphs and derived a purely graph-theoretic characterization of this problem. We draw attention to related works presented in [2] and [3], which were motivated by applications in two unrelated areas: network synthesis and routing in communications networks.

We now present the necessary background material on marked graphs. Though our presentation in this section is in terms of capacitated marked graphs, all of the definitions and results (Theorems 1 and 2) here are easy generalizations of those given in [4] and [5] for uncapacitated graphs. For terms not explicitly defined here, [5] may be consulted.

A *capacitated marked graph* is a marked graph $G=(V,E)$ in which a lower bound $L(e)$ and an upper bound $U(e)$ are specified on the token count $M(e)$ of edge $e \in E$, for all markings of G. A marking M is called *feasible* if and only if $L(e) \leq M(e) \leq U(e)$, $\forall e \in E$. The *enabling number* of a vertex $v \in V$ under a marking M of a capacitated marked graph G is defined as

$$\mu_v = \min\left\{ \min_{e \in E_v^-} \{M(e)-L(e)\},\ \min_{e \in E_v^+} \{U(e)-M(e)\}\right\} \tag{1}$$

where E_v^- and E_v^+ are the input and output edge sets of vertex v, respectively. This reduces to the usual definition for uncapacitated graphs when $L(e)=0$ and $U(e)=\infty, \forall e \in E$. A vertex v is *enabled* or *legally firable* under a marking M if its enabling number under M is greater than zero.

Let $C \subseteq E$ be a circuit of G and define an arbitrary orientation for C. Let C_+ and C_- denote the subsets of C consisting of all edges following and opposing that orientation, respectively. A *deadlock* of a capacitated marked graph G under a marking M is either

i) an edge $e \in E$ with $L(e)=U(e)$ or
ii) a circuit $C=C_+ \cup C_- \subseteq E$ such that either

$$M(e)=L(e), \forall e \in C_+ \quad \text{and} \quad M(e)=U(e), \forall e \in C_-$$

or

$$M(e)=U(e), \forall e \in C_+ \quad \text{and} \quad M(e)=L(e), \forall e \in C_-.$$

A capacitated marked graph is *live* under a marking M if each vertex $v \in G$ can be enabled through some legal firing sequence starting at M. Liveness of a capacitated marked graph is characterized in the following theorem.

Theorem 1: A capacitated marked graph G is live under a marking M if and only if G has no deadlocks under M. ■

Necessary and sufficient conditions for the reachability of capacitated marked graphs are given in the following theorem, where B_f is a fundamental circuit matrix of G.

Theorem 2 (Capacitated Reachability Theorem): Let M_0 and M_r be two feasible live markings of a marked graph G. Let $\Delta_M = M_r - M_0$. M_r and M_0 are mutually reachable if and only if $B_f \Delta_M = 0$. ■

A marked graph is called a *(0,1)-capacitated marked graph* if under any marking of the graph the number of tokens allowed on each edge of the graph is equal to 0 or equal to 1.

II. A Characterization of the Reachability Problem for (0,1)-Capacitated Marked Graphs

Consider a (0,1)-capacitated marked graph $G=(V,E)$ with a specified marking M_a. We define a *transformed graph* $G_a=(V_a,E_a)$ as follows.

i) G and G_a have the same underlying undirected graph.
ii) If we denote by e_a the edge of G_a corresponding to the edge e in G, then the orientation of e_a is opposite to that of e if $M_a(e)=1$; otherwise, e_a and e have the same orientation.

Theorem 3: A (0,1)-capacitated marked graph G^- has no deadlocks under M_a if and only if the transformed graph G_a is acyclic.

Proof: A deadlock in G is a circuit C which is of one of the following forms.

i) C is a token-free directed circuit of G.
ii) C is a directed circuit of G in which all of the edges are saturated (that is, the number of tokens on each edge of C is unity).
iii) C is a circuit of G in which all the edges in one direction are token free and those in the opposite direction are saturated.

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If we denote by C_a the circuit in G_a corresponding to a circuit in G, then it is easy to see from the definition of G_a that C_a is a directed circuit if and only if C is of one of the three forms mentioned above. Thus, G_a is acyclic if and only if G has no deadlocks. ■

Theorem 4: A (0,1)-capacitated marked graph G is live if and only if the transformed graph G_a is acyclic.

Proof: The proof follows from Theorem 3 and the fact that G is live if and only if it has no deadlocks (Theorem 1). ■

Consider any two markings M_a and M_b on a (0,1)-capacitated marked graph G. Let G_a and G_b denote the corresponding transformed graphs, respectively. As before, the edges in G_a and G_b which correspond to the edge $e \in G$ will be denoted by e_a and e_b, respectively. Similar notation will be used to denote the corresponding circuits. Let $\text{REV}(M_a, M_b) = \{e_a | e_a$ and e_b have opposing orientations$\}$.

Let C be a circuit in G and let us define an arbitrary circuit orientation on C. The same orientation will be assigned to the corresponding circuits C_a and C_b. Let

$$C_a^+ = \{ e_a | e_a \in \text{REV}(M_a, M_b),\ e_a \text{ follows the orientation of } C_a \}$$

$$C_a^- = \{ e_a | e_a \in \text{REV}(M_a, M_b),\ e_a \text{ opposes the orientation of } C_a \}.$$

In the following, the phrase "contribution of $\Delta_M(e) = M_b(e) - M_a(e)$" will refer to the quantity $\Delta_M(e)$ if e has the same orientation as that of C; otherwise, it will refer to the quantity $-\Delta_M(e)$.

Theorem 5: i) For any edge $e \in C$, the contribution of $\Delta_M(e)$ to C is positive if and only if $e_a \in C_a^+$; ii) For any edge $e \in C$, the contribution of $\Delta_M(e)$ to C is negative if and only if $e_a \in C_a^-$.

Proof of i):

Necessity: Consider any edge $e \in C$ such that the contribution of $\Delta_M(e)$ to C is positive. Then, either one of the following two cases should occur:

i) $M_a(e) = 0, M_b(e) = 1$ and e has the same orientation as that of C

ii) $M_a(e) = 1, M_b(e) = 0$ and e has an orientation opposite to that of C.

In both of these cases, $e_a \in C_a^+$.

Sufficiency: Consider any edge e such that $e_a \in C_a^+$. Then, either of the following two cases should occur:

i) $M_a(e) = 0, M_b(e) = 1$ and e has the same orientation as that of the circuit C

ii) $M_a(e) = 1, M_b(e) = 0$ and the orientation of e is opposite to that of C.

In both of these cases, the contribution of $\Delta_M(e)$ to C is positive.

Proof of statement ii) follows along the same lines as above. ■

We now present the main result of this paper.

Theorem 6: Consider a (0,1)-capacitated marked graph G. Let M_a and M_b be two live markings on G. M_b is reachable from M_a if and only if, for every circuit C in G,

$$|C_a^+| = |C_a^-|.$$

Proof: By Theorem 2, M_b is reachable from M_a if and only if the differential markings $\Delta_M(e)$'s satisfy KVL in G. By Theorem 5, the differential markings satisfy KVL in G if and only if, for each circuit C in G,

$$|C_a^+| = |C_a^-|. \quad ■$$

Note that a vertex v is enabled in G under the marking M_a if and only if v is a source in G_a. Furthermore, firing v in G corresponds to reversing the orientations of all the edges incident on v in G_a. Thus, we may define *firing a source* in G_a as the operation of reversing the orientations of all the edges incident on that source. In view of these observations, we can conclude from Theorem 6 that the acyclic graph G_a can be transformed into the acyclic graph G_b through a sequence of source firings if and only if the condition of this theorem is satisfied.

An easy consequence of Theorem 6 is stated next.

Corollary 6.1: If M_b is reachable from M_a on G, then $\text{REV}(M_a, M_b)$ is a cut in G_a.

Proof: First note that $\text{REV}(M_a, M_b)$ has an even number of common edges with every circuit in G_a. It is well known [1] that such a subset of edges is a cut in G_a. ■

At this point, we wish to draw attention to a related work on the concept of similarity introduced and studied in [2]. Consider a directed graph G. The operation of reversing the orientations of all the edges incident on a vertex v in G is called *switching* the vertex v in G. Note that, whereas firing is done only at a source vertex, switching is permitted at any vertex. Two directed graphs G_a and G_b having the same underlying undirected graph are *similar* if and only if one can be transformed into the other through a sequence of switchings. It has been proved in [2] that G_a and G_b are similar if and only if $\text{REV}(M_a, M_b)$ is a cut in G_a. Now, note from Corollary 6.1 that this necessary and sufficient condition for similarity is only a necessary condition for reachability of M_b and M_a. As in [2], we can also design, using depth-first-search, algorithms to test the reachability of M_b from M_a and to construct a sequence of source firings leading G_a to G_b.

Finally, we wish to point out that Gafni and Bertsekas [3] have introduced destination-oriented acyclic directed graphs in their study of a routing problem in communication networks. An acyclic directed graph (in short, ADG) with a special vertex (called the destination) is *destination oriented* if for every vertex there exists a directed path originating at this vertex and terminating at the destination. The problem considered in [3] is the following. Given a connected *destination-disoriented* ADG, transform it to a destination-oriented ADG by reversing the orientation of some of its edges. Gafni and Bertsekas have developed distributed algorithms for this problem. The fact that this problem is closely related to the problem considered in this paper suggests the possibility of designing efficient distributed algorithms for the reachability problem on (0,1)-capacitated marked graphs.

III. Summary

In this paper, we have presented a purely graph-theoretic characterization of the reachability problem on (0,1)-capacitated marked graphs. The relationship between this work and the results presented in [2] and [3] have been pointed out. This relationship suggests the possibility of designing efficient distributed algorithms for the reachability problem on (0,1)-capacitated marked graphs.

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An Improved Sufficient Condition for Absence of Limit Cycles in Digital Filters

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Abstract **—It is known that if the state transition matrix A of a digital filter structure is such that $D - A^{\dagger}DA$ is positive definite for some diagonal matrix D of positive elements, then all zero-input limit cycles can be suppressed. This paper shows that positive semidefiniteness of $D - A^{\dagger}DA$ is in fact sufficient. As a result, it is now possible to explain the absence of limit cycles in Gray–Markel lattice structures based only on the state-space viewpoint.**

I. A Property of the State Transition Matrix

Consider an IIR digital filter realization with the state-space description

$$x(n+1) = Ax(n) + Bu(n) \tag{1}$$

$$y(n) = Cx(n) + du(n) \tag{2}$$

where A is $N \times N$, B is $N \times 1$, C is $1 \times N$, and d is a scalar. In this paper, the input $u(n)$ is assumed to be zero. Fig. 1 shows a realistic model of the system, with quantizers in the feedback loop. The quantizers are such that each state variable is quantized independently of others:

$$x_k(n+1) = Q[w_k(n+1)]. \tag{3}$$

The operation $Q[x]$ represents magnitude-truncation arithmetic when $-1 \leqslant x < 1$ and 2's-complement overflow operation when x exceeds this range. Under this condition, it is well known [1] that if A satisfies[1]

$$A^{\dagger}A < I \tag{4}$$

then there are no self-sustained oscillations of either type (roundoff or overflow [2]) under zero input. Condition (4) is equivalent to saying that the singular values of A are strictly less than unity, or, in other words, that

$$V^{\dagger}A^{\dagger}AV < V^{\dagger}V \qquad \text{for every vector } V \neq 0. \tag{5}$$

Even though A is stable[2], (4) is in general not guaranteed unless

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[1]Superscript T denotes transposition and superscript $\dagger$ denotes transposed conjugation. The notation $P < Q$, where P and Q are Hermitian, denotes that $Q - P$ is positive definite; $P \leqslant Q$ denotes that $Q - P$ is positive semidefinite.

[2]We say that A is stable if all its eigenvalues are strictly inside the unit circle.

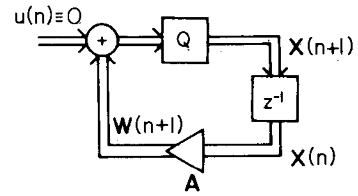

Fig. 1. The state recursion model with nonlinearity.

the state-space structure is appropriately chosen. The minimum-norm structures introduced in [1] automatically satisfy (4) because for such structures, the maximum eigenvalue of $A^{\dagger}A$ is equal to the maximum of the absolute values of the eigenvalues of A.

It is also well known in the literature that the cascaded lattice structures [4], [5] due to Gray and Markel are free from zero-input limit cycles as long as the quantizers are chosen as above. However, these structures do *not* satisfy (4). In fact, the normalized lattice structure has a state-space description [7], [8] as in (1), (2) satisfying

$$A^{\dagger}A + C^{\dagger}C = I. \tag{6}$$

Clearly, (6) implies

$$A^{\dagger}A \leqslant I. \tag{7}$$

Since C is a row vector, there exists a set of $N-1$ orthogonal vectors such that $V^{\dagger}A^{\dagger}AV = V^{\dagger}V$; hence, there are precisely $N-1$ singular values of A equal to unity, thereby violating (4). The fact that the lattice structures are free from limit cycles in spite of this can be explained based on the observation that the pair (C, A) "happens to be" completely observable [7], [8]. In this paper, we show that there is in fact a *fundamental* linear-algebraic reason why this should be so. A result is presented which shows that the complete observability in the lattice structures is by no means coincidental but is a consequence of a more basic result. This helps to obtain a formal, quantitative proof of certain useful claims made in an earlier paper [11].

The main result of this paper is the following.

Lemma 1: Let A be an $N \times N$ stable matrix satisfying (7). Then the following is true:

$$(A^{\dagger})^N A^N < I. \tag{8}$$

Condition (8) enables us to show that zero-input limit cycles will not be sustained.

Proof: First note that since (7) holds, we can always find a $p \times N$ matrix $\mathscr{C}$ such that

$$A^{\dagger}A + \mathscr{C}^{\dagger}\mathscr{C} = I. \tag{9}$$

The lemma is proved by establishing the following two properties:

Property 1: Condition (8) holds if and only if $\mathscr{C}$ satisfying (9) is such that $(\mathscr{C}, A)$ is *completely observable*.

Property 2: If A is stable, the pair $(\mathscr{C}, A)$ satisfying (9) is *necessarily* completely observable.

In order to prove Property 1, note that the pair $(\mathscr{C}, A)$ is completely observable if and only if the $pN \times N$ matrix P defined by

$$P = \begin{bmatrix} \mathscr{C} \\ \mathscr{C}A \\ \mathscr{C}A^2 \\ \vdots \\ \mathscr{C}A^{N-1} \end{bmatrix} \tag{10}$$

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