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# Synthesis of a class of resistive 3-port networks†

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Sufficient conditions are obtained for the synthesis of a class of 3-port resistive networks. The class is characterized by the property that all networks belonging to this class have a port configuration which is in the form of a sub-graph of a linear tree. Procedures for the synthesis of such networks are given. A significant feature of these realization procedures is that for any realization a large number of equivalent realizations can be obtained.

## 1. Introduction

The problem of synthesis of resistive $n$-port networks is considered an outstanding one in network theory (Newcomb 1966). Whereas synthesis of $n$-port networks with $n+1$ nodes is completely known, the problem of synthesis with more than $n+1$ nodes is yet to be solved. Recently an approach to this problem was presented (Thulasiraman 1967). This approach was later used to obtain certain sufficient conditions and a procedure for the synthesis of a class of 3-port resistive networks (Easwaran 1968). In this paper we explain and illustrate the results obtained by Easwaran (1968).

Some of the theorems and definitions which are used in subsequent discussions are stated below. Unless otherwise stated the notation used in this paper is the same as that used by Cederbaum (1965) and Thulasiraman and Murti.

*Definition* 1: *Potential factor*

The potential factor $K_{ij}$ in an $n$-port network is defined as the potential of the positive reference terminal of port $j$ with respect to the negative reference terminal of port $i$ when port $i$ is excited with a source of unit voltage and all the other ports short-circuited.

It follows from definition that $K_{ii}$ is unity.

*Theorem* 1

The element $c_{ij}$ of the matrix $C$ is equal to the voltage appearing across the edge corresponding to column $j$ when port $i$ is excited with a source of unit voltage and all the other ports short-circuited.

*Theorem* 2

If $G$ be the edge conductance matrix of a given resistive $n$-port network and $C$ its modified cut-set matrix, then (i) $CGC_2'=0$ and (ii) $CGC_1'=Y$.

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*Theorem* 3

Let $C$ be the modified cut-set matrix of a given resistive $n$-port network $N_1$. If any real diagonal matrix $G_2$ satisfies the equation $CG_2C_2'=0$, then the modified cut-set matrix of an $n$-port network $N_2$ with $G_2$ as its edge conductance matrix and having identical port and edge configuration and orientation as $N_1$ is also equal to $C$.

*Theorem* 4

A sufficient condition for the proper parallel connection of two resistive $n$-port networks is that their modified cut-set matrices be equal when their corresponding edges and ports are similarly oriented. This condition is also necessary if positive resistances only are permitted.

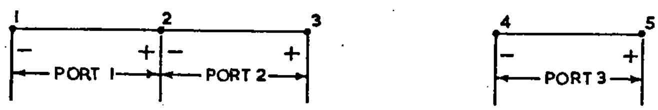
Proofs of the above theorems were found by Thulasiraman and Murti.

## 2. Philosophy of the approach

The procedure to be followed in deriving the sufficient conditions for the synthesis of 3-port resistive networks having the port structure shown in fig. 1 is briefly explained in this section.

Let $Y$ be the given third-order real symmetric matrix which has to be realized as the short-circuit conductance matrix of a network having the port structure shown in fig. 1:

Fig. 1



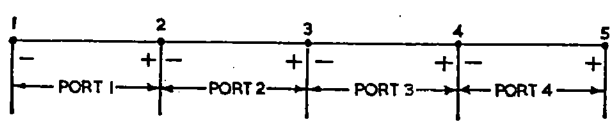

$$Y=\begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix}. \tag{1}$$

Following Guillemin (1961) the above matrix is augmented to one of order 4 by inserting a null row and column as shown below:

$$[Y]_{\text{exp}}=\begin{bmatrix} y_{11} & y_{12} & 0 & y_{13} \\ y_{21} & y_{22} & 0 & y_{23} \\ 0 & 0 & 0 & 0 \\ y_{31} & y_{32} & 0 & y_{33} \end{bmatrix}. \tag{2}$$

$[Y]_{\text{exp}}$ is then realized as the short-circuit conductance matrix of a 4-port network $N_1$ having the port configuration shown in fig. 2.

Fig. 2



If $g_{ij}$ refers to the conductance of the edge joining the vertices $i$ and $j$, then the column matrix $\{g_1\}$ of the conductances of the edges in $N_1$ can be obtained as follows using the procedure given by Guillemin (1961):

$$\{g_1\} = \begin{bmatrix} y_{11}-y_{12} \\ y_{12} \\ -y_{13} \\ y_{13} \\ y_{22}-y_{12} \\ y_{13}-y_{23} \\ y_{23}-y_{13} \\ y_{23} \\ -y_{23} \\ y_{33} \end{bmatrix}. \tag{3}$$

It may be observed that $N_1$ considered as a 3-port network having the port configuration specified in fig. 1 has a short-circuit conductance matrix equal to $Y$ and its modified cut-set matrix does not exist. Also some of the conductances of $N_1$ will be negative. Consider next a 3-port network $N_2$ having the specified port configuration and the zero matrix as its short-circuit conductance matrix. Let $\{g_2\}$ represent the column matrix of the edge conductances of $N_2$. Then the parallel combination of $N_1$ and $N_2$, denoted as $N_3$, will have an edge conductance matrix $\{g_3\} = \{g_1\} + \{g_2\}$ and its short-circuit conductance matrix will be equal to $Y$. Hence, if a network $N_2$ can be found so that $N_3$ contains only non-negative conductances, then the synthesis of $Y$ will be complete.

We next proceed to explain a method for finding a network whose short-circuit conductance matrix is the zero matrix. From theorems 2 and 3 any diagonal matrix $G_2$ satisfying the equations:

$$CG_2C_2' = 0, \tag{4}$$

$$CG_2C_1' = 0, \tag{5}$$

represents the edge conductance matrix of a 3-port network $N_2$ having (i) its modified cut-set matrix equal to $C$ and (ii) its short-circuit conductance matrix equal to zero.

From theorem 1 it is clear that the modified cut-set matrix of any network can be written in terms of the potential factors. For a 3-port network having the port configuration shown in fig. 1, the potential factors $K_{12} = 1$, $K_{21} = 0$ and $K_{31} = K_{32}$. Let $K_{13} = P$, $K_{23} = Q$ and $K_{31} = K_{32} = R$. It is to be noted that $P$, $Q$ and $R$ are greater than zero but less than unity. The modified cut-set matrix $C$ can then be written as:

$$C = \begin{array}{c} \begin{array}{cccccccccc} 12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \end{array} \\ \begin{bmatrix} 1 & 1 & P & P & 0 & P-1 & P-1 & P-1 & P-1 & 0 \\ 0 & 1 & Q & Q & 1 & Q & Q & Q-1 & Q-1 & 0 \\ 0 & 0 & -R & -R+1 & 0 & -R & -R+1 & -R & -R+1 & 1 \end{bmatrix} \end{array}. \tag{6}$$

In the above matrix the column headed by $ij$ refers to the edge joining vertices $i$ and $j$.

The fundamental cut-set matrix $C_0$ of the 3-port network with respect to the tree consisting of the edges $e_{12}$, $e_{23}$, $e_{34}$ and $e_{45}$ can be written as $C_0 = [C_1/C_2]$. The sub-matrix $C_1$ corresponds to the edges $e_{12}$, $e_{23}$ and $e_{45}$ and the sub-matrix $C_2$ corresponds to the edge $e_{34}$. $C_1$ and $C_2$ are obtained as:

$$\begin{array}{cc} & \begin{array}{cccccccccc} 12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \end{array} \\ C_1 = & \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \\ C_2 = & \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}. \end{array} \tag{7}$$

Denoting the column matrix of edge conductance of network $N_2$ as $\{g_2\}$ and taking into account the symmetry of eqn. (5), eqns. (4) and (5) can be written as:

$$\begin{bmatrix} 0 & 0 & P & P & 0 & P-1 & P-1 & P-1 & P-1 & 0 \\ 0 & 0 & Q & Q & 0 & Q & Q & Q-1 & Q-1 & 0 \\ 0 & 0 & -R & -R+1 & 0 & -R & -R+1 & -R & -R+1 & 0 \end{bmatrix} \{g_2\} = 0, \tag{8}$$

$$\begin{bmatrix} 1 & 1 & P & P & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & P & P & 0 & P-1 & P-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & P & 0 & 0 & P-1 & 0 & P-1 & 0 \\ 0 & 1 & Q & Q & 1 & Q & Q & 0 & 0 & 0 \\ 0 & 0 & 0 & Q & 0 & 0 & Q & 0 & Q-1 & 0 \\ 0 & 0 & 0 & -R+1 & 0 & 0 & -R+1 & 0 & -R+1 & 1 \end{bmatrix} \{g_2\} = 0. \tag{9}$$

The solution of the above sets of equations is obtained as:

$$\{g_2\} = \begin{bmatrix} \dfrac{-K(1-P)(P-Q)}{R(1-R)} \\ \dfrac{-K(1-P)Q}{R(1-R)} \\ \dfrac{K(1-P)}{R} \\ \dfrac{K(1-P)}{1-R} \\ \dfrac{-KQ(P-Q)}{R(1-R)} \\ \dfrac{K(P-Q)}{R} \\ \dfrac{K(P-Q)}{1-R} \\ \dfrac{KQ}{R} \\ \dfrac{KQ}{1-R} \\ -K \end{bmatrix}. \tag{10}$$

If it is possible to choose suitable values for $P$, $Q$, $R$ and $K$ so that a $\{g_2\}$ leading to a $\{g_3\}=\{g_1\}+\{g_2\}$ containing only non-negative elements can be obtained, then the synthesis of the given $Y$ matrix will be complete.

## 3. Sufficient conditions and synthesis procedure

Sufficient conditions for the realization of a third-order real symmetric matrix as the short-circuit conductance matrix of a 3-port network having the port structure shown in fig. 1 can be derived using the approach described in the previous section. These sufficient conditions are given below in the form of theorems.

*Theorem* 5

A real symmetric third-order matrix $Y=[y_{ij}]$ with positive values of $y_{12}$, $y_{13}$, $y_{23}$ and $(y_{13}-y_{23})$ can be realized as the short-circuit conductance matrix of a 3-port network having the port structure shown in fig. 1 if the following conditions are satisfied:

$$y_{11} > y_{13}+y_{12}-y_{23},$$

$$y_{33} \geq y_{11},$$

$$y_{12} > y_{23}.$$

*Theorem* 6

A real symmetric third-order matrix $Y=[y_{ij}]$ with positive values of $y_{12}$ and $y_{13}$ and with negative value of $y_{23}$ can be realized as the short-circuit conductance matrix of a 3-port network having the port structure shown in fig. 1 if the following conditions are satisfied:

$$y_{12} \geq |y_{23}|,$$

$$y^1(y_{22}-y_{12})-y_{12}y_{23} \geq y_{12}y_{13} \geq y_{22}y_{13}+y^2{}_{12}-y^1y_{12},$$

where $y^1$ is the smaller of $y_{33}$ and $y_{11}$.

Proofs of the above theorems were found by Easwaran (1968).

It can be observed that a third-order matrix with a sign pattern other than the two considered above can be changed to one having one of them either by interchanging the ports or by changing the polarity of some of the ports. Hence the sufficient conditions given above can be applied to other sign patterns also after the latter are transformed to any one of the two above-mentioned sign patterns.

Next we give the steps involved in the realization of a real symmetric third-order matrix $Y$ satisfying the sufficient conditions mentioned in either of the above two thoerems.

*Case* 1: *Short-circuit conductance matrix $Y$ with positive values of $y_{12}$, $y_{13}$, $y_{23}$ and $(y_{13}-y_{23})$.*

*Step* 1: Choice of the constant $k$:

Let $h$ be the higher of the two values:

$$\frac{y_{12}(y_{13}-y_{23})}{(y_{11}-y_{12})(y_{22}-y_{12})+(y_{13}-y_{23})y_{12}} \quad \text{and} \quad \frac{y_{23}}{y_{22}}.$$

Choose any positive $k$ satisfying:

$$h \leq k < y_{12}/y_{22}. \tag{11}$$

*Step* 2: Calculation of the value of the constant $K$:

For any value of the constant $k$ chosen in step 1, the constant $K$ is given by:

$$K = k y_{22} y_{11} / y_{12}. \tag{12}$$

*Step* 3: Calculation of potential factors:

Choose any value of the potential factor $R$ satisfying the following:

$$\frac{1 - y_{12}/y_{11}}{1/k - y_{12}/y_{11}} \leq R \leq \frac{k y_{22}(y_{11}/y_{12} - 1) - (y_{13} - y_{23})}{y_{11} - k y_{22} - (y_{13} - y_{23})}. \tag{13}$$

It may be noted that for each value of $k$ in the interval given in step 1, a large number of values of $R$ in turn can be chosen.

For a particular choice of $R$ the potential factors $Q$ and $P$ are given by:

$$Q = \frac{y_{12}}{y_{11}}(1 - R) \tag{14}$$

and

$$R = 1 - \frac{y_{12} R}{y_{22} k}. \tag{15}$$

*Step* 4: Edge conductances $\{g_3\}$:

Choosing $K$, $P$, $Q$ and $R$ as described above $\{g_2\}$ can be calculated using eqn. (10). From the given matrix $Y$, $\{g_1\}$ can be obtained, from eqn. (3). Then the column matrix $\{g_3\}$ of the conductances of the edges in network $N_3$ realizing the matrix $Y$ is given by $\{g_3\} = \{g_1\} + \{g_2\}$.

*Case* 2: *Short-circuit conductance matrix with positive values of $y_{12}$ and $y_{13}$ and negative value of $y_{23}$.*

*Step* 1: Choice of the constant $k$:

Let $h$ be the higher of the two values:

$$\frac{[y_{22} y_{13} + y_{12}(y_{12} - y_{13})](y_{22} - y_{12})}{y_{12}^2(|y_{23}| + y_{13})} \quad \text{and} \quad 1.$$

Choose any positive value $k$ satisfying:

$$h \leq k \leq \frac{y_{33}(y_{22} - y_{12})}{y_{12}(|y_{23}| + y_{13})}. \tag{16}$$

*Step* 2: Calculation of the value of the constant $K$:

For any value of $k$ chosen in step 1, the constant $K$ is given by:

$$K = k y_{12} \frac{|y_{23}| + y_{13}}{y_{22} - y_{12}}. \tag{17}$$

*Step* 3: Calculation of potential factors:

Choose any value of the potential factor $R$ satisfying the following:

$$\frac{\dfrac{|y_{23}| + y_{13}}{K}}{\dfrac{|y_{23}| + y_{13}}{K} + 1 - \dfrac{y_{12}}{K}} \leq R \leq \frac{\dfrac{y_{22} - y_{12}}{y_{12}}}{\dfrac{y_{22} - y_{12}}{y_{12}} + 1 - \dfrac{y_{12}}{K}}. \tag{18}$$

The values of the potential factors $P$ and $Q$ are given by:

$$\left.\begin{aligned} P &= R, \\ Q &= \frac{y_{12}R}{K}. \end{aligned}\right\} \tag{19}$$

*Step* 4: Edge conductances $\{g_3\}$:

Choosing $K$, $P$, $Q$ and $R$ as described above, $\{g_2\}$ can be calculated using eqn. (10). From the given matrix $Y$ and using eqn. (3), $\{g_1\}$ can be obtained. Then the column matrix $\{g_3\}$ of the conductances of the edges in network $N_3$ realizing the matrix $Y$ is given by $\{g_3\} = \{g_1\} + \{g_2\}$.

*Example*

Consider the following matrix $Y'$:

$$Y' = \begin{bmatrix} 8 & -6 & 3 \\ -6 & 10 & -4 \\ 3 & -4 & 20 \end{bmatrix}.$$

Changing the polarities of port 2 and interchanging ports 1 and 2 leads to:

$$Y = \begin{bmatrix} 10 & 6 & 4 \\ 6 & 8 & 3 \\ 4 & 3 & 20 \end{bmatrix}.$$

It may be observed that the matrix $Y$ satisfies the conditions mentioned in theorem 5. Hence, realization of $Y$ can be accomplished using the procedure given under case 1, as shown below.

*Step* 1: According to eqn. (11) any positive $k$ satisfying $0{\cdot}428 \leq k < 0{\cdot}75$ is to be chosen. Let $k = 0{\cdot}6$.

*Step* 2: Using eqn. (12) the constant $K$ is calculated for the above chosen value of $k$ as $8{\cdot}0$.

*Step* 3: According to eqn. (13) the value of the potential factor $R$ is to be chosen such that it satisfies:

$$0{\cdot}374 \leq R \leq 0{\cdot}523.$$

Let $R = 0{\cdot}5$.

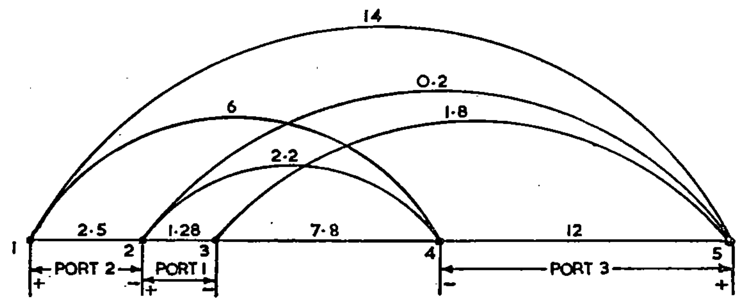
The potential factors $Q$ and $P$ are calculated respectively using eqns. (14) and (15) as $Q = 0{\cdot}3$ and $P = 0{\cdot}375$.

*Step* 4: The column matrix of the edge conductances of the network realizing $Y$ is obtained as:

$$\{g_3\} = \begin{bmatrix} 2{\cdot}5 \\ 0 \\ 6 \\ 14 \\ 1{\cdot}28 \\ 2{\cdot}2 \\ 0{\cdot}2 \\ 7{\cdot}8 \\ 1{\cdot}8 \\ 12 \end{bmatrix}.$$

The network which realizes the given matrix $Y'$ is shown in fig. 3. The edge conductance values in mhos are indicated on the figure.

Fig. 3




## 4. Conclusions

Sufficient conditions are developed for the synthesis of a class of 3-port resistive networks. An important feature of the realization procedure given in this paper is that for a given matrix satisfying the relevant sufficient conditions after suitable changes in the signs and positions of rows and columns, a large number of equivalent networks can be obtained by choosing different values of $R$. This has advantages when the conductance values of certain edges are preassigned. The method can be readily used for the synthesis of 3-port 2-element-kind networks if the residue matrices satisfy the sufficient conditions stated.

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