SIMPLE SENSITIVITY FORMULAS IN TERMS OF IMMITTANCE PARAMETERS

Indexing terms: Sensitivity analysis, Active circuits

Simple formulas are derived for computing the sensitivities of a network function with respect to its network elements for a lumped/distributed active network. These formulas are in terms of immittance parameters of an augmented network. It is shown that these formulas may be used directly for the computation of higher-order sensitivities.

Recently,[1] formulas for 2nd-order sensitivities of a network function have been obtained in terms of partial derivatives of element currents or voltages with respect to element immittances. However, such formulas cannot easily be used to determine 2nd-order sensitivities. Hence it is desirable to obtain formulas for these sensitivities in terms of the immittance parameters of the network. In this letter, we first obtain such formulas for 1st-order sensitivities, and then indicate the procedure used to obtain formulas for higher-order sensitivities.

Consider an n-port lumped/distributed active network N. Let each internal element be a 1-port or a 2-port element. Let there be E internal ports in N. If these are also treated as external ports, then an $(n+E)$-port network denoted by N^* can be obtained from N. Let the first n ports of N^* correspond to the external ports of N, and let N^A denote the adjoint of N (see Reference 2).

Consider any internal element of N having an admittance matrix Y_e. Let the two ports of this element correspond to the ports λ and μ of N^*, and let

$$Y_e = \begin{bmatrix} Y_{\lambda\lambda} & Y_{\lambda\mu} \\ Y_{\mu\lambda} & Y_{\mu\mu} \end{bmatrix} \qquad (1)$$

If V_e and ψ_e denote the vector of voltages across the ports of an internal element in N and N^A, respectively, it can be shown[3] that

$$\frac{\partial Z_{ij}}{\partial Y_{\lambda\mu}} = -\psi_e^t \frac{\partial Y_e}{\partial Y_{\lambda\mu}} V_e = -\psi_\lambda V_\mu \qquad (2)$$

Now

$$V_\mu = Z^*_{\mu j} I_j = Z^*_{\mu j}, \qquad (3)$$

since I_j is unity, and

$$\psi_\lambda = Z^*_{i\lambda} \phi_i = Z^*_{i\lambda} \qquad (4)$$

since ϕ_i is unity, where the symbol * denotes the parameters of N^*. In obtaining eqn. 4, the fact that the impedance matrix of the adjoint of a network is the transpose of the impedance matrix of the network is utilised.[4, 5]

Thus, from eqns. 2, 3 and 4,

$$\frac{\partial Z_{ij}}{\partial Y_{\lambda\mu}} = -Z^*_{i\lambda} Z^*_{\mu j} \qquad (5)$$

Eqn. 5 may be used directly to find the sensitivity components $(\partial Z_{ij}/\partial Y_{\lambda\mu})$ corresponding to lumped passive elements, as well as gyrators and v.c.t.s. These are given in Table 1. Now, for a c.v.t., Y_e does not exist, and hence eqn. 2 cannot be used. However, Z_e exists with $Z_{\lambda\lambda} = Z_{\lambda\mu} = Z_{\mu\mu} = 0$ and $Z_{\mu\lambda} = r_{\mu\lambda}$. Then[3]

$$\frac{\partial Z_{ij}}{\partial r_{\mu\lambda}} = \phi_\mu I_\lambda = \frac{\psi_\lambda}{r_{\mu\lambda}} \frac{V_\mu}{r_{\mu\lambda}} = \frac{Z^*_{i\lambda} Z^*_{\mu j}}{r^2_{\mu\lambda}}$$

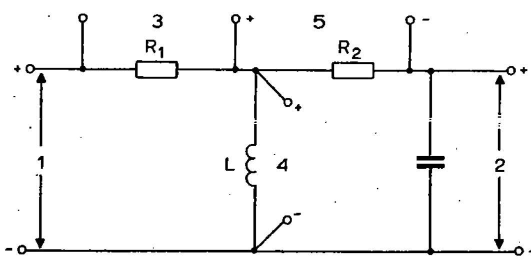

Fig. 1 *Network for the example*

Table 1

Elements	Parameters	Sensitivity components
Conductance	G_k	$\frac{\partial Z_{ij}}{\partial G_k} = -Z^*_{ik} Z^*_{kj}$
Elastance •	$\Gamma_k = \frac{1}{L_k}$	$\frac{\partial Z_{ij}}{\partial \Gamma_k} = -\frac{1}{s} Z^*_{ik} Z^*_{kj}$
Capacitance	C_k	$\frac{\partial Z_{ij}}{\partial C_k} = -sZ^*_{ik} Z^*_{kj}$
Gyrator	$Y_e = \begin{bmatrix} 0 & \beta_{\lambda\mu} \\ \beta_{\mu\lambda} & 0 \end{bmatrix}$	$\frac{\partial Z_{ij}}{\partial \beta_{\lambda\mu}} = -Z^*_{i\lambda} Z^*_{\mu j}$ $\frac{\partial Z_{\cdot j}}{\partial \beta_{\mu\lambda}} = -Z^*_{i\mu} Z^*_{\lambda}$
V.C.T.	$Y_e = \begin{bmatrix} 0 & 0 \\ \delta_{\mu\lambda} & 0 \end{bmatrix}$	$\frac{\partial Z_{ij}}{\partial \delta_{\mu\lambda}} = -Z^*_{i\mu} Z^*_{\lambda j}$
C.V.T.	$Z_e = \begin{bmatrix} 0 & 0 \\ r_{\mu\lambda} & 0 \end{bmatrix}$	$\frac{\partial Z_{ij}}{\partial g_{\mu\lambda}} = -Z^*_{i\lambda} Z^*_{\mu j}$
$\overline{RC}$ tapered lines	$Y_e = \begin{bmatrix} Y_{\lambda\lambda} & Y_{\lambda\mu} \\ Y_{\mu\lambda} & Y_{\mu\mu} \end{bmatrix}$	$\frac{\partial Z_{ij}}{\partial G_{0i}} = -\left\{\frac{\partial Y_{\lambda\lambda}}{\partial G_{0i}} Z^*_{i\lambda} Z^*_{\lambda j} + \frac{\partial Y_{\lambda\mu}}{\partial G_{0i}} Z^*_{i\lambda} Z^*_{\mu j} + \frac{\partial Y_{\mu\lambda}}{\partial G_{0i}} Z^*_{i\mu} Z^*_{\lambda j} + \frac{\partial Y_{\mu\mu}}{\partial G_{0i}} Z^*_{i\mu} Z^*_{\mu j}\right\}$
		$\frac{\partial Z_{ij}}{\partial C_{0i}} = -\left\{\frac{\partial Y_{\lambda\lambda}}{\partial C_{0i}} Z^*_{i\lambda} Z^*_{\lambda j} + \frac{\partial Y_{\lambda\mu}}{\partial C_{0i}} Z^*_{i\lambda} Z^*_{\mu j} + \frac{\partial Y_{\mu\lambda}}{\partial C_{0i}} Z^*_{i\mu} Z^*_{\lambda j} + \frac{\partial Y_{\mu\mu}}{\partial C_{0i}} Z^*_{i\mu} Z^*_{\mu j}\right\}$
Lossless tapered lines	$Y_e = Y_{\overline{LC}i} \begin{bmatrix} \psi_{\lambda\lambda} & \psi_{\lambda\mu} \\ \psi_{\mu\lambda} & \psi_{\mu\mu} \end{bmatrix}$	$\frac{\partial Z_{ij}}{\partial Y_{\overline{LC}i}} = -\{\psi_{\lambda\lambda} Z^*_{i\lambda} Z^*_{\lambda j} + \psi_{\lambda\mu} Z^*_{i\lambda} Z^*_{\mu j} + \psi_{\mu\lambda} Z^*_{i\mu} Z^*_{\lambda j} + \psi_{\mu\mu} Z^*_{i\mu} Z^*_{\mu j}\}$

or

$$\frac{\partial Z_{ii}}{\partial g_{\mu\lambda}} = -Z^*_{i\lambda} Z^*_{\mu j} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $g_{\mu\lambda} = 1/r_{\mu\lambda}$.

Consider next an RC tapered line as the internal element of N. Let it be described by

$$\bar{R}_i = R_{0i} f(x) \quad \bar{C}_i = C_{0i} g(x) \quad a \leqslant x \leqslant b \quad . \quad . \quad (7)$$

where $\bar{R}_i$ and $\bar{C}_i$ are the series resistance per unit length and the shunt capacitance per unit length, respectively.

It should be pointed out that eqn. 5 cannot be used directly to compute the sensitivity of Z_{ij} with respect to R_{0i} or C_{0i}, since all the elements of Y_e for an $\overline{RC}$ line depend on R_{0i} and C_{0i}. Now

$$\frac{\partial Z_{ij}}{\partial G_{0i}} = \frac{\partial Z_{ij}}{\partial Y_{\lambda\lambda}} \frac{\partial Y_{\lambda\lambda}}{\partial G_{0i}} + \frac{\partial Z_{ij}}{\partial Y_{\lambda\mu}} \frac{\partial Y_{\lambda\mu}}{\partial G_{0i}} + \frac{\partial Z_{ij}}{\partial Y_{\mu\lambda}} \frac{\partial Y_{\mu\lambda}}{\partial G_{0i}} + \frac{\partial Z_{ij}}{\partial Y_{\mu\mu}} \frac{\partial Y_{\mu\mu}}{\partial G_{0i}} \quad (8)$$

where $G_{0i} = 1/R_{0i}$.

Using eqn. 5, eqn. 8 may be written as

$$\frac{\partial Z_{ij}}{\partial G_{0i}} = -\left\{\frac{\partial Y_{\lambda\lambda}}{\partial G_{0i}} Z^*_{i\lambda} Z^*_{\lambda j} + \frac{\partial Y_{\lambda\mu}}{\partial G_{0i}} Z^*_{i\lambda} Z^*_{\mu j} + \frac{\partial Y_{\lambda\mu}}{\partial G_{0i}} Z^*_{i\mu} Z^*_{\lambda j} + \frac{\partial Y_{\mu\mu}}{\partial G_{0i}} Z^*_{i\mu} Z^*_{\mu j}\right\} \quad (9)$$

Similarly, $\partial Z_{ij}/\partial C_{0i}$ can be obtained, and is given in Table 1.

Consider an $\overline{LC}$ tapered line described by

$$\bar{L}_i = L_{0i} f(x) \quad \bar{C}_i = C_{0i} g(x) \quad a \leqslant x \leqslant b \quad . \quad . \quad (10)$$

where $\bar{L}_i$ and $\bar{C}_i$ are the series inductance and shunt capacitance per unit length. If the 'characteristic impedance' is defined as

$$Z_{LCi} = \sqrt{(L_{0i}/C_{0i})} = 1/Y_{\overline{LC}i}$$

then the admittance matrix for the LC line can be written as

$$Y_e = Y_{\overline{LC}i} \begin{bmatrix} \psi_{\lambda\lambda} & \psi_{\lambda\mu} \\ \psi_{\mu\lambda} & \psi_{\mu\mu} \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where $\psi_{\lambda\lambda}$ etc. are all functions of $(L_{0i} C_{0i})$ only. Assuming that $(L_{0i} C_{0i})$ is constant, the sensitivity of Z_{ij} with respect to $Y_{\overline{LC}i}$ may be obtained using eqns. 5 and 11, and is given in Table 1.

Consider next the voltage transfer T_V between the jth and ith ports. Then

$$T_V = \frac{Z_{ji}}{Z_{ii}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\frac{\partial T_V}{\partial Y_{\lambda\mu}} = \frac{\partial}{\partial Y_{\lambda\mu}}\left(\frac{Z_{ji}}{Z_{ii}}\right) = \frac{1}{Z_{ii}} \frac{\partial Z_{ji}}{\partial Y_{\lambda\mu}} + \frac{Z_{ji}}{Z^2_{ii}}\left(\frac{-\partial Z_{ii}}{\partial Y_{\lambda\mu}}\right)$$

$$= \frac{1}{Z_{ii}}(-Z^*_{\mu i} Z^*_{j\lambda}) + \frac{Z_{ji}}{Z^2_{ii}}(Z^*_{\mu i} Z^*_{i\lambda})$$

$$= -Z^*_{j\lambda} T^*_{\mu i} + Z^*_{i\lambda} T^*_{ji} T^*_{\mu i} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Dual expressions may be obtained for the admittance and current transfer functions.

Formulas for 2nd-order sensitivity components can now be obtained in a straightforward manner with the help of the formulas for 1st-order ones. For example, if p_1 and p_2 correspond to the admittances of R, L or C, then

$$\frac{\partial^2 Z_{ij}}{\partial p_2 \partial p_1} = \frac{\partial}{\partial p_2}\left(\frac{\partial Z_{ij}}{\partial p_1}\right) = \frac{\partial}{\partial p_2}(-Z^*_{i1} Z^*_{1j})$$

$$= Z^*_{i1} Z^*_{12} Z^*_{2j} + Z^*_{21} Z^*_{i2} Z^*_{1j} \quad . \quad . \quad . \quad . \quad (14)$$

Higher-order sensitivity components may be determined by the repeated use of Table 1.

Example: Consider the 2-port network shown in Fig. 1. The 2nd-order sensitivity of the transfer impedance Z_{12} with respect to the admittances of branches 2 and 4 is evaluated, using eqn. 14, as

$$\frac{\partial^2 Z_{12}}{\partial Y_2 \partial Y_4} = Z^*_{14} Z^*_{42} Z^*_{22} + Z^*_{12} Z^*_{24} Z^*_{42} \quad . \quad (15)$$

where Z^*_{14}, Z^*_{42}, Z^*_{12}, Z^*_{24}, Z^*_{42} and Z^*_{22} are the open-circuit impedance parameters of the augmented 5-port network. These parameters can be easily evaluated and are given below:

$$Z^*_{14} = \frac{Y_2 + Y_5}{Y_4 Y_5 + Y_5 Y_2 + Y_2 Y_4} \quad . \quad . \quad . \quad . \quad . \quad (16a)$$

$$Z^*_{42} = Z^*_{24} = \frac{Y_5}{Y_4 Y_5 + Y_5 Y_2 + Y_2 Y_4} \quad . \quad . \quad . \quad (16b)$$

$$Z^*_{12} = \frac{Y_5}{Y_4 Y_5 + Y_5 Y_2 + Y_2 Y_4} \quad . \quad . \quad . \quad . \quad . \quad (16c)$$

$$Z^*_{22} = \frac{Y_4 + Y_5}{Y_4 Y_5 + Y_5 Y_2 + Y_2 Y_4} \quad . \quad . \quad . \quad . \quad . \quad (16d)$$

where

$$\left.\begin{aligned} Y_2 &= sC \\ Y_4 &= \frac{1}{sL} = \frac{\Gamma}{s} \\ Y_5 &= G_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Substituting eqns. 16 and 17 into eqn. 15, we obtain

$$\frac{\partial^2 Z_{12}}{\partial \Gamma \partial C} = \frac{2G_2{}^3 s^3 + G_2 s^2(s^2 CG_2 + sC\Gamma + \Gamma G_2)}{(s^2 CG_2 + sC\Gamma + \Gamma G_2)^3} \quad . \quad . \quad (18)$$

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