written as

$$X = Tx'. \tag{7}$$

Utilizing the relation in (7) in the respective state model of the networks under consideration, we can obtain the following relations:

$$AT = TA' \tag{8}$$

$$AB_2 + B_1 = T(A'B_2' + B_1') \tag{9}$$

$$CT = C' \tag{10}$$

where A', B_1', B_2', and $C' = C$ are the coefficient matrices in the state model of the network in Fig. 1(b).

The entries of the transformation matrix T, for continuously equivalent networks, however, are the function of the real parameter x and, if $A(0)$ is the A matrix of the original network and $A(x)$ is the A matrix of an equivalent network for a given parameter x, then the transformation matrix $T(x)$ satisfies the relation [6], [13]

$$A(0)T(x) = T(x)A(x). \tag{11}$$

If x is increased by a small amount, say Δx, we may approximately take

$$T(x + \Delta x) \cong T(x)[I + K\Delta x] \tag{12}$$

where K is assumed to be a square matrix of order n. Substituting (12) into (11) yields

$$\frac{d}{dx} A(x) = A(x)K - KA(x) \tag{13}$$

and the solution, as given in [11] and [13] is

$$A(x) = e^{-Kx}A(0)e^{Kx} \tag{14}$$

which is in similar form as the forms obtained by the equivalent networks theories of Howitt and Schoeffler [3]. Comparing (14) with (11) we obtain

$$T(x) = e^{Kx}. \tag{15}$$

Hence, it is clear from the above discussion that for a fixed value of x, the transformation matrix $T(x) = e^{Kx}$ transforms the original network in Fig. 1(a) into its equivalent network in Fig. 1(b).

From the relation in (10) it can be shown easily that the transformation $T(x)$ is of the form

$$T(x) = \begin{bmatrix} t_{11}(x) & t_{12}(x) \\ 0 & 1 \end{bmatrix}. \tag{16}$$

Substituting (16) into (15) we obtain

$$K = \begin{bmatrix} k_{11} & k_{12} \\ 0 & 0 \end{bmatrix} \tag{17}$$

and

$$t_{11} = e^{k_{11}x}$$
$$t_{12} = \frac{k_{12}}{k_{11}} [e^{k_{11}x} - 1]. \tag{18}$$

Then substituting (17) into (13) we obtain

$$\frac{d}{dx}\begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix} = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix}\begin{bmatrix} k_{11} & k_{12} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} k_{11} & k_{12} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix}. \tag{19}$$

The whole problem now is to obtain the solution of (19) subject to initial conditions such that at $x = 0$ the $A(0)$ matrix of the circuit in Fig. 1(a) results, and at $x = x_0$ the $A(x_0)$ matrix of the circuit in Fig. 1(b) results. After obtaining the matrix $A(x_0)$ and utilizing the relations obtained in (8) and (9), all the element values of the network in Fig. 1(b) can be determined.

Example

Let the element values of the network in Fig. 1(a) be given as

$$r_1 = \tfrac{1}{2}\,\Omega, \quad L_1 = -1\text{ H}, \quad L_3 = 2\text{ H}$$
$$r_3 = 2\,\Omega, \quad L_2 = 2\text{ H}, \quad C_2 = \tfrac{1}{2}\text{ F}$$

where $L = L_1 + L_2 = 1$ H and $n = -L_2/L = -2$.

Let $k_{11} = 1$ and $k_{12} = -4$. Then the solution of (19) at $x_0 = l_n\ 7/5$ yields $a_{11}(x_0) = -3/5$, $a_{12}(x_0) = 29/35$, $a_{21}(x_0) = -7/5$, and $a_{22}(x_0) = -3/20$. Also using the relations obtained from (8) and (9), the element values of the equivalent network in Fig. 1(b) can be determined as

$$r_1' = 39/14\,\Omega, \quad L_1' = -15/7\text{ H}, \quad L_3' = 6/7\text{ H}$$
$$r_3' = -2/7\,\Omega, \quad L_2' = -10/7\text{ H}, \quad C_2' = -7/2\text{ F}$$
$$L' = -25/7\text{ H}, \quad n' = -2/5$$

where $r_1 + r_3 = r_1' + r_3'$ and $r_3c_2 = r_3'c_2'$.

IV. Conclusion

In general, a state model representation

$$R = \langle A, (B_1, B_2, \cdots, B_{n+1}), C\rangle$$

where A, B_i, and C are, respectively, $(n\times n)$, $(n\times m)$, and $(m\times n)$ matrices, can be transformed into a standard form $R = \langle \hat{A}, \hat{B}, \hat{C}\rangle$ from which the minimality of the representation R can be determined [10].

If the representation R of any two networks is minimal, there exists a transformation matrix T between the state vectors of these networks that can be interpreted as the continuously equivalent network transformation between these networks [10]. In this correspondence, the approach makes use of the above results, which are mentioned in detail in [10].

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A Sufficient Condition for the Synthesis of the K-Matrix of n-Port Networks

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In a recent paper Lempel and Cederbaum [1] have discussed the analysis and synthesis of the K-matrix of n-port networks. The results of Lempel and Cederbaum include procedures for the determination of the port configuration pertinent to a given K-matrix and for the synthesis of a given K-matrix by a resistive n-port network having more than $(n+1)$-nodes. Subsequently, Reddy and Thulasiraman [2] have presented an alternate approach for K-matrix synthesis. The procedures given in [1] as well as in [2] require the solution of a linear program. In this correspondence we establish, using the results of [2], a simple sufficient condition for the synthesis of the K-matrix of an n-port network having more than $(n+1)$-nodes and

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having a specified port configuration. We first summarize briefly the results established in [2] and [3] that will be useful in later discussions.

Definition 1—Network of Departure

An n-port network with more than $(n+1)$-nodes is called a network of departure N_d, with respect to a real symmetric matrix Y, if its cutset admittance matrix is equal to

$$\begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}$$

where the rows of Y correspond to the port edges.

Definition 2—Padding n-Port Network

An n-port network is called a padding n-port network N_p if its short-circuit admittance matrix is equal to zero.

It can be shown that any n-port network N can be considered as the parallel combination of a network of departure N_d and a padding n-port network N_p and that N_d and N_p for a given n-port network N are unique. [3]

Consider a resistive n-port network N. Let the port configuration T of N be in p parts: $T_1, T_2 \cdots T_p$. It may be assumed, without any loss of generality, that each T_i is a Lagrangian tree. The set of vertices of T_i will be denoted by $i_0, i_1, i_2, \cdots, i_{n_i}$. The mth port of T_i will be denoted by P_{i_m}. i_0 and i_m are the negative and positive reference terminals of P_{i_m}. Denoting by $g_{i_k j_m}$ the conductance connecting vertices i_k and j_m, we then define $S_{i_k j}$ and S_{ij} as follows:

$$S_{i_k j} = \sum_{m=0}^{nj} g_{i_k j_m}, \qquad j \neq i$$

$$S_{ij} = \sum_{k=0}^{ni} S_{i_k j} = \sum_{m=0}^{nj} S_{j_m i}, \qquad j \neq i.$$

Subscripts d and p will be used to refer to the quantities of the network of departure N_d and the padding n-port network N_p of N. It can be shown that $(S_{ij})_p = S_{ij}$ and $(S_{i_k j})_p = S_{i_k j}$.

The potential factor matrix $K = [k_{ij}]$ of an n-port network N is defined as the $n \times n$ matrix where k_{ij}, called the potential factor of port j with respect port i, is the potential of the positive reference terminal of port j with respect to the negative reference terminal of port i when port i is excited with unit voltage and all the other ports are short circuited. After rearranging its rows and columns, the matrix K can be partitioned as follows:

$$K = \begin{bmatrix} K_{11} & K_{12} \cdots K_{1p} \\ K_{21} & K_{22} \cdots K_{2p} \\ \cdots & \cdots \\ \cdots & \cdots \\ K_{p1} & K_{p2} \cdots K_{pp} \end{bmatrix}$$

where a) each submatrix K_{ii}, $i = 1, 2, \cdots, p$, has entries comprising 1 and 0 only and is uniquely fixed by T_i; b) each entry in a submatrix $K_{ij}, j \neq i$, is less than unity and greater than zero except in degenerate cases; c) all entries in any row of each $K_{ij}, j \neq i$, are equal.

Let a typical column of K_{ij}, $j \neq i$, be denoted by $\boldsymbol{K}_{ij}$. Let $k_{i_k j}$ denote the voltage of the set of ports in T_j when port P_{i_k} is excited with a source of unit voltage and all the other ports are short circuited. We note that $k_{i_k j}$ is equal to some element of the kth row of $K_{ij}, j \neq i$.

The following equations have been obtained in [2]:

$$S_{i_k j} = S_{ij} k_{i_k j} + \sum_{\substack{m=1 \\ m \neq i,j}}^{p} S_{jm}(k_{i_k j} - k_{i_k m}), \qquad 1 \leq i,\ j \leq p,\ j \neq i;\ 1 \leq k \leq n_i$$

$$S_{i_0 j} = S_{ij} - \sum_{k=1}^{ni} S_{i_k j} \tag{1}$$

$$(g_{i_k j_m})_p = S_{j_m i} k_{i_k j} + \sum_{\substack{r=1 \\ r \neq i,j}}^{p} S_{j_m r}(k_{i_k j} - k_{i_k r}), \qquad 1 \leq i,\ j \leq p,\ j \neq i;\ 1 \leq m \leq n_j;\ 1 \leq k \leq n_i$$

$$(g_{i_0 j_m})_p = S_{j_m i} - \sum_{k=1}^{ni} g_{j_m i_k}$$

$$(g_{i_k i_m})_p = -\sum_{\substack{j=1 \\ j \neq i}}^{p} S_{i_m j} k_{i_k j}, \qquad 1 \leq i \leq p;\ 1 \leq k \leq n_i;\ 0 \leq m \leq n_i;\ k \neq m. \tag{2}$$

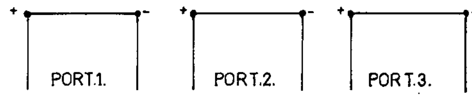

Fig. 1.

The following results form the basis of the procedure given in [2] for K-matrix synthesis.

1) The potential factor matrix of N_p is the same as that of N.

2) N_p is the padding network of some n-port network N containing no negative conductances if and only if all $(S_{i_k j})_p$ are nonnegative. Thus synthesis of the K-matrix reduces to the determination of nonnegative S_{ij} so that $S_{i_k j}$, as obtained by (1), are nonnegative.

We now prove the following Theorem.

Theorem

A real matrix $K = [k_{ij}]$ can be realized as the potential factor matrix of a resistive n-port network having a port configuration $T = T_1 U T_2 U \cdots T_p$, with each T_i a Lagrangian tree, if a) each K_{ii} contains 0's and 1's only and is realizable by T_i; b) all columns of each K_{ij}, $j \neq i$, are identical; c) $\boldsymbol{K}_{ij} = \boldsymbol{K}_{im}$, $1 \leq i \leq p-1$, $1 \leq j$, $m \leq p$, $j, m \neq i$; d) the sum of all entries of each $\boldsymbol{K}_{ij}$, $j \neq i$, is less than unity.

Proof: We prove the Theorem by establishing the existence of a set of nonnegative values for S_{ij} which result in nonnegative $S_{i_k j}$. Condition a) of the theorem ensures that the required network will have the port configuration T and condition b), as stated earlier, is a property of the K-matrix.

From condition c) it follows that $k_{i_k j} = k_{ikm}$, $1 \leq i \leq p-1$; $1 \leq j$, $m \leq p$. Hence, from (1) we get

$$S_{i_k j} = S_{ij} k_{i_k j}, \qquad 1 \leq i \leq p-1;\ 1 \leq j \leq p$$

$$S_{i_0 j} = \left(1 - \sum_{k=1}^{ni} k_{i_k j}\right) S_{ij}. \tag{3}$$

In view of condition d), we can conclude from the above equation that any set of nonnegative values for S_{ij}, $1 \leq i, j \leq p$, will result in nonnegative $S_{i_k j}$ for all $1 \leq i \leq p-1$, $1 \leq j \leq p$, $j \neq i$.

Consider next the following equation:

$$S_{p_i j} = S_{pj} k_{p_i j} + \sum_{\substack{m=1 \\ m \neq p,j}}^{p} S_{jm}[k_{p_i j} - k_{p_i m}]; \qquad 1 \leq j \leq p-1;\ 1 \leq i \leq n_p$$

$$S_{p_0 j} = S_{pj} - \sum_{k=1}^{np} S_{p_k j}, \qquad 1 \leq j \leq p-1. \tag{4}$$

Since each $S_{p_i j}$ should be greater than zero, we can get from the above a lower bound for S_{pj} in terms of the remaining S_{ij}. Let such a lower bound be denoted by L_j.

Then any set of nonnegative values for S_{ij}, $1 \leq i$, $j \leq p-1$, and a nonnegative value for S_{pj} such that $S_{pj} \geq L_j$ will result in nonnegative values for all $S_{i_k j}$. Hence, the Theorem.

Example

Let it be required to realize the following matrix as the K-matrix of a 3-port resistive network:

$$K = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/3 & 1 & 1/3 \\ 1/2 & 1/4 & 1 \end{bmatrix}.$$

It can be shown that the port structure of the required 3-port network is as shown in Fig. 1.

Using (3) and the given matrix K, the following relations are obtained:

$$S_{1_1 2} = \tfrac{1}{2} S_{12} \qquad S_{1_0 2} = \tfrac{1}{2} S_{12}$$
$$S_{1_1 3} = \tfrac{1}{2} S_{13} \qquad S_{1_0 3} = \tfrac{1}{2} S_{13}$$
$$S_{2_1 1} = \tfrac{1}{3} S_{12} \qquad S_{2_0 1} = \tfrac{2}{3} S_{12}$$
$$S_{2_1 3} = \tfrac{1}{3} S_{23} \qquad S_{2_0 3} = \tfrac{2}{3} S_{23}. \tag{5}$$

The above relations indicate that any set of nonnegative values for S_{ij}, $1 \leq i$, $j \leq 3$, will result in nonnegative $S_{i_k j}$ for all $1 \leq i \leq 2$ and $1 \leq j \leq 3$, $j \neq i$.

The following relations are then obtained using (4):

$$S_{3_1 1} = \tfrac{1}{2} S_{13} + \tfrac{1}{4} S_{12} \qquad S_{3_0 1} = \tfrac{1}{2} S_{13} - \tfrac{1}{4} S_{12}$$
$$S_{3_1 2} = \tfrac{1}{4} S_{23} - \tfrac{1}{4} S_{12} \qquad S_{3_0 2} = \tfrac{3}{4} S_{23} + \tfrac{1}{4} S_{12}. \tag{6}$$

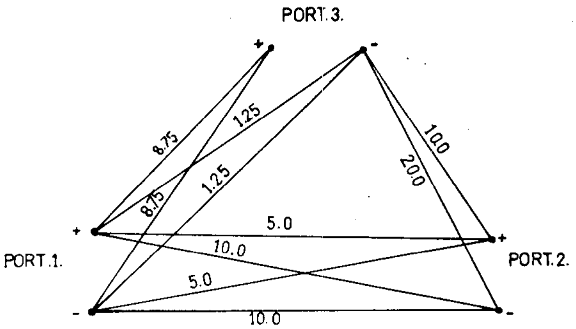

Fig. 2. A sufficient condition for the synthesis of the K-matrix of n-port networks.

From the above equation it can be seen that $S_{3_1 1}$, $S_{3_0 1}$, $S_{3_1 2}$, and $S_{3_0 2}$ will be nonnegative if

$$S_{13} \geq \tfrac{1}{2} S_{12}$$

and

$$S_{23} \geq S_{12}. \tag{7}$$

One set of nonnegative values for S_{ij} satisfying (7) is as follows:

$$S_{12} = 30 \quad S_{13} = 20 \quad S_{23} = 30.$$

Using the above values of S_{ij} and (5) and (6), all $S_{i_k j}$ are then evaluated. Using these $S_{i_k j}$ and (14) of [3], the conductances of a 3-port network N having the K-matrix can be obtained. The network N is shown in Fig. 2.

In conclusion, we wish to point out that no necessary and sufficient conditions, besides the procedures given in [1] and [2], are available for testing the realizability of K-matrices of resistive n-port networks having more than $(n+1)$-nodes. However, such conditions are available in the special cases of $(n+2)$-node and $(n+3)$-node n-port networks [3], [4]. The sufficient condition given in Theorem 1 is applicable for any resistive n-port network with $p>1$.

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On the Synthesis of the Two-Port in Bode's Variable Equalizer with Pivots

EUGEN SUHĂREANU AND MARIA SUHĂREANU

I. Introduction

Bode's variable equalizer was studied in a recent correspondence [1] under the particular condition $\alpha-\alpha_0>0$ at all real frequencies for $0<\rho\leq 1$. The effect of this condition on the variable equalizer is that it cannot provide a prescribed curve $\alpha-\alpha_0$ with pivots.[1]

Under the general condition $\alpha-\alpha_0 \gtrless 0$ with the insertion loss $\alpha \geq 0$ for either $0<\rho\leq 1$ or $-1\leq\rho<0$, Bode's variable equalizer has pivots, because at some frequencies $\alpha-\alpha_0$ becomes zero. For this reason it is often named equalizer with pivots. It can be obtained, under certain conditions on the curve C, with a minimum phase two-port Q, and under other conditions on the curve C, with a nonminimum phase two-port Q.

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[1] The frequencies for which $\alpha-\alpha_0=0$ at $\rho\neq 0$ are named pivots [2].

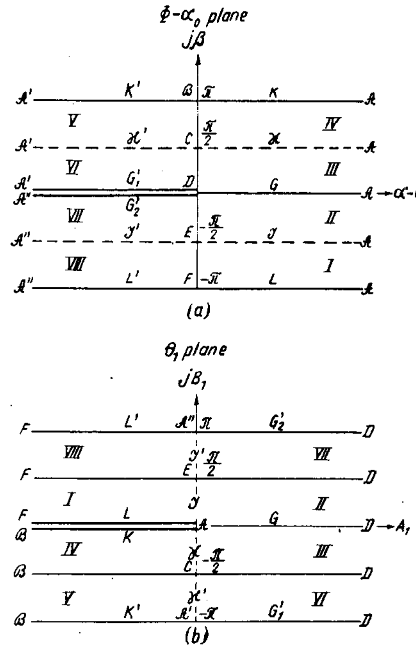

Fig. 1. (a) Strip P having the branch cut along the negative real axis. (b) Image of strip P in Fig. 1(a) under (1).

The purpose of this correspondence is to find, in the case of the variable equalizer with pivots, the set of necessary and sufficient conditions on the curve C under which Q results in a minimum phase two-port. This set of necessary and sufficient conditions is deduced with the help of the conformal mapping of the domain in which the variable equalizer with pivots is defined.

The significance of the notations used is found in [1].

II. Conformal Mapping of Strip P of the $\phi-\alpha_0$ Plane onto the θ Plane

We rewrite the relation (4) of [1]:

$$\theta = A + jB = \frac{1}{2}\ln \rho u + \frac{1}{2}\ln \frac{e^{\phi-\alpha_0}+1}{e^{\phi-\alpha_0}-1} = \frac{1}{2}\ln \rho u + \frac{1}{2}\theta_1 \tag{1}$$

where

$$\theta_1 = A_1 + jB_1 = \ln \frac{e^{\phi-\alpha_0}+1}{e^{\phi-\alpha_0}-1}$$

for

$$0 < \rho \leq 1$$

or

$$\theta = A + jB = \frac{1}{2}\ln |\rho| u + \frac{1}{2}\ln \frac{1+e^{\phi-\alpha_0}}{1-e^{\phi-\alpha_0}} = \frac{1}{2}\ln |\rho| u + \frac{1}{2}\theta_1 \tag{2}$$

where

$$\theta_1 = A_1 + jB_1 = \ln \frac{1+e^{\phi-\alpha_0}}{1-e^{\phi-\alpha_0}}$$

for

$$-1 \leq \rho < 0.$$

Since θ may be deduced from θ_1 by a translation and a multiplication, we study θ with the help of θ_1.

The functions (1) and (2) are multiple-valued, having the critical points $\phi-\alpha_0=jk\pi (k=0, \pm 1, \pm 2, \cdots)$ and $\phi-\alpha_0=\infty$. Let there be the infinite strip $-\pi\leq\beta\leq\pi$, designated P, in the $\phi-\alpha_0$ plane. Each