will contain all the directed circuits of G containing the vertex v_i. Therefore, if each $e_{ii}M_{ii}(i = 1, 2, \cdots, n)$ has a term in common, it can only mean that there exists a directed circuit containing all the vertices of the graph G, which, by definition, is a directed Hamilton circuit. If there is no such common term, it simply means that there is no directed circuit containing all the vertices.

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Comments "On Equivalence of Resistive n-Port Networks"

Cederbaum[1] has recently proposed a method of realization of the short-circuit admittance matrix of a resistive n-port network, based on considerations of network equivalence. The purpose of this correspondence is to point out certain properties of the modified cutset matrix C used by him and some limitations of his procedure which restrict its use in a practical realization problem.

Cederbaum's method starts with the realization of the short-circuit admittance matrix Y in the form of a $2n$-node network in which the alternate branches of a linear tree are identified as the n accessible ports. The nodes are numbered consecutively starting from one end vertex of the linear tree. The nodes numbered $2i - 1$ and $2i$ form the port i. The orientation of any edge (i, j) joining the nodes i and j with $j > i$ and having the conductance g_{ij} is from j to i. The network configuration between any two ports is a symmetric lattice in which the conductances of the cross arms or the series arms are zero, depending on the sign of the transfer admittance. The conductances of some of the edges shunting the ports are negative if the matrix Y is nondominant but are all non-negative if Y is dominant. Thus the network between any two ports i and j (with $j > i$) has the form shown in Fig. 1, where the following relations hold:

$$\begin{aligned} g_{2i-1,2i} &= y_{ii} - \sum_{\substack{k=1\\k\neq i}}^{k=n} |y_{ik}| \\ g_{2j-1,2j} &= y_{jj} - \sum_{\substack{k=1\\k\neq j}}^{k=n} |y_{jk}| \end{aligned} \tag{1}$$

$$\begin{aligned} g_{2i-1,2j-1} = g_{2i,2j} &= 0, && \text{if} \quad y_{ij} \geq 0 \\ &= 2\,|y_{ij}|, && \text{if} \quad y_{ij} < 0 \\ g_{2i-1,2j} = g_{2i,2j-1} &= 0, && \text{if} \quad y_{ij} \leq 0 \\ &= 2\,y_{ij}, && \text{if} \quad y_{ij} > 0. \end{aligned}$$

Let the edges of this network be grouped so that the ith group of edges contains all edges (i, j) where $j > i$ and let the mth edge in the ith group be $(i, i + m)$. Now consider the fundamental cutset matrix of the network with the ith row corresponding to the branch $(i, i + 1)$ of the linear tree and the columns corresponding to edges grouped and ordered as specified above. Then the submatrix of the fundamental cutset matrix corresponding to the ith group of columns is given by

$$\begin{array}{ccc} & \longleftarrow (2n-i)\text{ columns} \longrightarrow & \\ \begin{matrix}\uparrow\\ (2n-1)\\ \text{rows}\\ \downarrow\end{matrix} & \begin{bmatrix} 0 & 0 & \cdots\cdots\cdots & 0 & 0 \\ \cdot & \cdot & \cdots\cdots\cdots & \cdot & \cdot \\ 0 & 0 & \cdots\cdots\cdots & 0 & 0 \\ 1 & 1 & \cdots\cdots\cdots & 1 & 1 \\ 0 & 1 & \cdots\cdots\cdots & 1 & 1 \\ \cdot & \cdot & \cdots\cdots\cdots & \cdot & \cdot \\ 0 & 0 & \cdots\cdots\cdots & 1 & 1 \\ 0 & 0 & \cdots\cdots\cdots & 0 & 1 \end{bmatrix} & \begin{matrix}\uparrow\\ (i-1)\text{ rows}\\ \downarrow\\ \\ \\ \\ \\ \\ \end{matrix} \end{array} . \tag{2}$$

The fundamental cutset matrix is next transformed so that the odd-numbered rows in the above occur first in order and then the even-numbered rows. Let this matrix be

$$C_0 = \left[\frac{C_1}{C_2}\right], \tag{3}$$

where C_1 corresponds to the alternate branches of the linear tree which are identified as the n accessible ports and C_2 corresponds to the remaining branches.

Let G be the diagonal matrix of edge conductances with column ordering corresponding to that of the fundamental cutset matrix C_0. Let

$$C_0\,G\,C_0' = \left[\begin{array}{c|c} C_1\,G\,C_1' & C_1\,G\,C_2' \\ \hline C_2\,G\,C_1' & C_2\,G\,C_2' \end{array}\right] = \left[\begin{array}{c|c} Y_{11} & Y_{12} \\ \hline Y_{21} & Y_{22} \end{array}\right]. \tag{4}$$

Then the short-circuit admittance matrix Y of the n-port network is given by

$$\begin{aligned} Y &= Y_{11} - Y_{12}Y_{22}^{-1}Y_{21} \\ &= (C_1 - Y_{12}Y_{22}^{-1}C_2)G(C_1 - Y_{12}Y_{22}^{-1}C_2)' \\ &= CGC', \end{aligned} \tag{5}$$

where C is called the modified cutset matrix. It may be noted that the modified cutset matrix is in general dependent on both the network topology and the edge conductances g_{ij}.

Cederbaum shows that if dG is an infinitesimally small diagonal matrix satisfying the equation,

$$C\,dG\,C^r = 0, \tag{6}$$

then $(G + dG)$ is the diagonal matrix of edge conductances of a new network realizing the short-circuit admittance matrix Y. However, this procedure for generating equivalent networks will be practically useful only if the change in G is not constrained to be infinitesimally

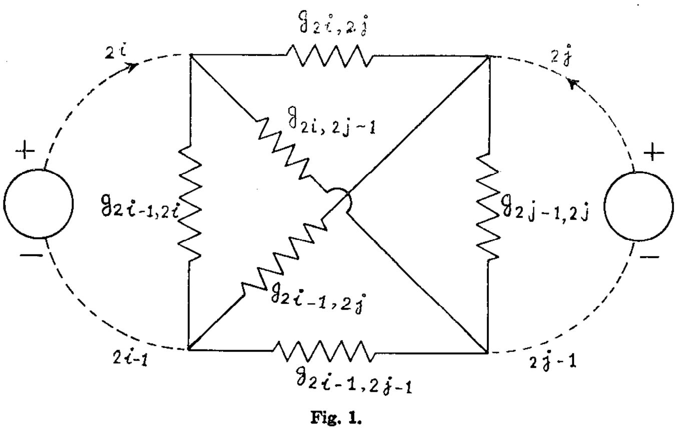

Fig. 1.

Manuscript received November 1, 1966; revised April 26, 1967.

small. Let us therefore consider the case of a diagonal matrix ΔG, which is not required to be an infinitesimal increment. Any ΔG satisfying the equation,

$$C\,\Delta G\,C' = 0 \tag{7}$$

will lead to a new realization of Y with edge conductances as in $(G + \Delta G)$, provided the modified cutset matrix of the new network is the same as the modified cutset matrix C of the original network. This is clear from the following.

$$\begin{aligned}\begin{array}{l}\text{Short-circuit admittance}\\ \text{matrix of the new network}\end{array} &= C(G + \Delta G)C' \\ &= CGC' + C\,\Delta GC' = CGC' = Y.\end{aligned} \tag{8}$$

Now the network taken for the starting point in Cederbaum's procedure has the property that if port i is excited with a voltage source V and all the other ports short-circuited, then the short-circuited ports are all at a potential $\frac{1}{2}V$ with respect to terminal $2i$–1. It can be shown that the modified cutset matrix of such a network assumes the following standard form independent of edge conductances.[2] If C is partitioned as

$$C = [C^1 \,\vdots\, C^2 \,\vdots\, \text{- - -} \,\vdots\, C^{2i-1} \,\vdots\, C^{2i} \,\vdots\, \text{- - -} \,\vdots\, C^{2n-1}], \tag{9}$$

where the submatrices C^{2i-1} and C^{2i} have columns corresponding to the $(2i$–$1)$th and the $2i$th groups of edges, respectively, then

$$[C^{2i-1} \,\vdots\, C^{2i}] = n \text{ rows} \begin{array}{c} \uparrow \\ \Bigg| \\ \Bigg| \\ \downarrow \end{array} \begin{array}{c} \begin{array}{cc} \leftarrow (2n-2i+1) \text{ columns} \rightarrow & \leftarrow (2n-2i) \text{ columns} \rightarrow \end{array} \\ \left[\begin{array}{ccccccc:cccccc} 0 & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0 & 0 & 0\cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot\cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0\cdots & 0 & 0 & 0 & 0 & 0 & 0\cdots & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\cdots & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}\cdots & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0\cdots & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0\cdots & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2}\cdots & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2}\cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot\cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0\cdots & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0\cdots & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \end{array} \begin{array}{l} \uparrow \\ (i-1) \text{ rows} \\ \downarrow \\ \\ \\ \\ \\ \\ \end{array} . \tag{10}$$

It can also be shown that a $2n$-node n-port network having the above form for its modified cutset matrix should contain a symmetric lattice between every pair of ports.[2]

Reverting to the procedure of generating equivalent networks, it is therefore seen that the new network with edge conductances $(G + \Delta G)$ must also contain symmetric lattices to ensure that the modified cutset matrix is not altered. This implies that ΔG itself corresponds to a network with a symmetric lattice between every pair of ports because the original network has this property.

Let us next consider the solution of (7). If the given Y is not dominant, the standard realization procedure according to (1) does not yield positive conductances for all the edges. We therefore seek a solution of ΔG in (7) such that $(G + \Delta G)$ corresponds to a proper realization. The matrix $D = C\Delta GC'$ is symmetric and each of its elements is a linear combination of the diagonal elements of ΔG, i.e., the conductances Δg_{ij}. Equating the distinct elements of this matrix to zero, we get $\frac{1}{2}n(n+1)$ homogeneous equations in Δg_{ij} which are $n(2n-1)$ in number. Let us next arrange these equations into groups so that the ith group corresponds to all elements d_{ij} of the matrix D, where $j = i+1, i+2, i+3, \cdots, n$. Thus there will be $(n-1)$ such groups of equations. Let the nth group contain all equations corresponding to the diagonal elements d_{ii} with $i = 1, 2, \cdots, n$. The equations in matrix form will be

$$[S]\{\Delta g\} = 0, \tag{11}$$

where the row ordering of the column matrix $\{\Delta g\}$ is the same as in ΔG. The submatrix S_i of S containing the ith group of rows is given by

$$S_i = \begin{array}{c} \uparrow \\ (n-i) \\ \text{rows} \\ \downarrow \end{array} \begin{array}{c} \begin{array}{cc} \leftarrow \begin{array}{c}\text{columns relating to}\\ (2i-1)\text{th group}\\ \text{of edges}\end{array} \rightarrow . & \leftarrow \begin{array}{c}\text{columns relating to}\\ 2i\text{th group}\\ \text{of edges}\end{array} \rightarrow \end{array} \\ \left[\begin{array}{c:c:ccccccc:ccccc:c:c} 0\cdots 0 & \text{- - -} & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & 0\cdots & 0 & 0 & \frac{1}{4} & -\frac{1}{4} & 0 & 0\cdots 0 & 0 & \text{- - -} & 0 \\ 0\cdots 0 & \text{- - -} & 0 & 0 & 0 & -\frac{1}{4} & \frac{1}{4}\cdots & 0 & 0 & 0 & 0 & \frac{1}{4} & -\frac{1}{4}\cdots 0 & 0 & \text{- - -} & 0 \\ \hdashline 0\cdots 0 & \text{- - -} & 0 & 0 & 0 & 0 & 0\cdots & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0\cdots\frac{1}{4} & -\frac{1}{4} & \text{- - -} & 0 \end{array}\right] \end{array} . \tag{12}$$

We now consider the mth row in S_i and observe the following.

1) The elements in the $2m$th column of the $(2i$–$1)$th group of columns and the $2m$th column of the $2i$th group of columns are both $(-\frac{1}{4})$. These two columns correspond, respectively, to the unknowns $\Delta g_{2i-1,2i-1+2m}$ and $\Delta g_{2i,2i+2m}$.

2) The elements in the $(2m+1)$th column of the $(2i-1)$th group of columns and the $(2m-1)$th column of the $2i$th group of columns are both $(\frac{1}{4})$. These columns correspond to $\Delta g_{2i-1,2i+2m}$ and $\Delta g_{2i,2i+2m-1}$.

3) The elements in all other columns are zero. So the mth equation in the ith group of equations can be written as

$$\begin{aligned}&(1/4)(\Delta g_{2i-1,2i-1+2m} + \Delta g_{2i,2i+2m}) \\ &\qquad = (1/4)(\Delta g_{2i-1,2i+2m} + \Delta g_{2i,2i+2m-1}).\end{aligned} \tag{13}$$

Generalizing this result for all i and making the substitution $j = i + m$, we have

$$\Delta g_{2i-1,2j-1} + \Delta g_{2i,2j} = \Delta g_{2i-1,2j} + \Delta g_{2i,2j-1}$$
$$\text{for } i = 1, 2, \cdots, n-1 \text{ and } j = i+1, i+2, \cdots, n. \tag{14}$$

It is already noted that to preserve the form of the modified cutset matrix, the network corresponding to ΔG must have a symmetric lattice between every pair of ports i and j; i.e.,

$$\Delta g_{2i-1,2j-1} = \Delta g_{2i,2j}$$
and
$$\Delta g_{2i,2j-1} = \Delta g_{2i-1,2j}. \tag{15}$$

We have from (14) and (15),

$$\Delta g_{2i-1,2j-1} = \Delta g_{2i-1,2j} = \Delta g_{2i,2j-1} = \Delta g_{2i,2j} = \Delta k_{ij}. \tag{16}$$

That is, all the additional conductances Δg of the lattice network between every pair of ports i and j have to be equal so that $(G + \Delta G)$

may correspond to a network realizing the matrix Y. This is also clear from Fig. 2, where the additional transmission from port i to port j due to ΔG is zero because of the balanced bridge configuration, resulting in no change in y_{ij} in going from G to $(G + \Delta G)$. Since in the network corresponding to G, two of the four edges joining the ports have zero conductance, it is also additionally required that Δk_{ij} be non-negative to ensure that $(G + \Delta G)$ leads to a proper realization.

The submatrix S_n corresponding to the nth group of equations in (11) is given by

$$S_n = [S_n^1 \mid S_n^2 \mid --- \mid S_n^{2i-1} \mid S_n^{2i} \mid --- \mid S_n^{2n-1}],$$

where

$$\begin{array}{ccc}
 & \begin{array}{cc} \leftarrow \text{ columns relating to } \rightarrow & \leftarrow \text{ columns relating to } \rightarrow \\ (2i-1)\text{th group} & 2i\text{th group} \\ \text{of edges} & \text{of edges} \end{array} & \\
[S_n^{2i-1} \mid S_n^{2i}] = \begin{array}{c} \uparrow \\ n \\ \text{rows} \\ \downarrow \end{array} & \left[\begin{array}{cccccccc:ccccccc}
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{4} & \frac{1}{4} \\
0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \cdots & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \cdots & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \cdots & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & \cdots & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & \cdots & \frac{1}{4} & \frac{1}{4}
\end{array}\right] & \begin{array}{c} \uparrow \\ (i-1) \text{ rows} \\ \downarrow \\ \\ \\ \\ \end{array}
\end{array} \quad . \qquad (17)$$

To start with, at least one $g_{2i-1,2i}$ is negative if the given Y matrix is not dominant and we wish to choose $\Delta g_{2i-1,2i}$ such that $g_{2i-1,2i} + \Delta g_{2i-1,2i}$ is non-negative. It may be observed that the entries in the first column of the $(2i-1)$th group of columns in matrix S_n are all zeros except in the ith row where it is 1. The entries in this column are coefficients of $\Delta g_{2i-1,2i}$ in the group of equations

$$[S_n]\{\Delta g\} = \{0\}. \qquad (18)$$

If $(g_{2i-1,2i} + \Delta g_{2i-1,2i})$ which has to be non-negative for a proper realization replaces $\Delta g_{2i-1,2i}$ as the unknown in this matrix equation, then on the right-hand side of (18) we have to replace the zero in the ith row by $g_{2i-1,2i}$. If such a change of variable is made for all the n equations in (18), there results the following:

$$[S_n]\{g'\} = \{g\}. \qquad (19)$$

In this set of equations, the conductances represented by $\{g'\}$ are the incremental conductances Δk_{ij} of the edges connecting one port with another and the new values of conductances $g_{2i-1,2i} + \Delta g_{2i-1,2i}$ of edges shunting the ports. $\{g\}$ represents the original values of the latter. We require a solution of $\{g'\}$ yielding non-negative values for all its components. This is clearly impossible if the given short-circuit admittance matrix is not dominant, as the coefficient matrix S_n contains only positive terms and $\{g\}$ contains at least one negative component. Hence a proper realization of Y is not possible by this procedure if Y is not dominant.

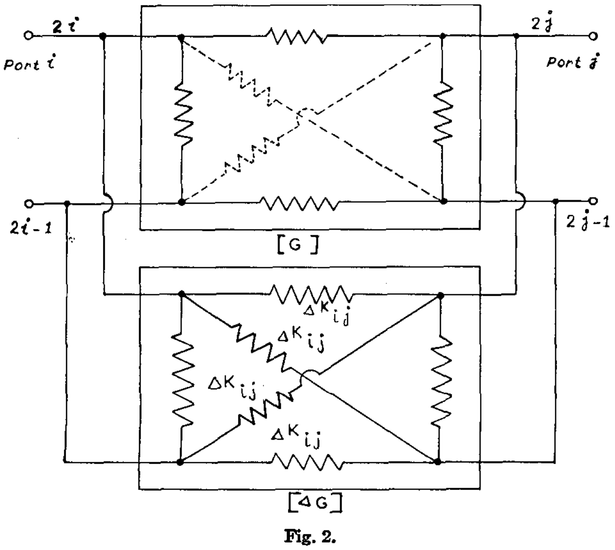

Fig. 2.

If, however, we choose any solution for ΔG satisfying the constraints in (7) and (15), then the new network corresponding to $(G + \Delta G)$ is a realization of Y (but not a proper realization) and has the modified cutset matrix of the form in (10). If the procedure discussed so far is repeated with this new network to generate a third equivalent network, a proper realization will not be found possible once again. It is therefore to be concluded that if the network of Fig. 1 is taken as the network of departure and equivalent n-port networks obtained by the solution of (7), the procedure can yield networks with all positive elements only if the given Y matrix is dominant. It cannot however lead to a proper realization of a nondominant short-circuit admittance matrix.

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A Generalized SubGraph—The *k* Seg

INTRODUCTION

In applied graph theory a subgraph is often defined for particular application and later shown to be a special case of a more general class of subgraphs. The tree is a special case of the k tree and the star is a cutset which is in turn a seg. In this correspondence the generalization of the star is carried an additional step with the definition of the k seg. The factor which has made the seg and its subclasses significant is that a generalization of the Kirchhoff current law is applicable to the currents in the elements of a seg.[3]

Unlike the seg, which is either a cutset or an element disjoint union of cutsets, k segs are not in general members of some other class of subgraphs nor can they be obtained by simple set theoretic operations from other subgraphs. They do, however, have many properties which are analogous to those of segs and cutsets.

Manuscript received February 28, 1967.