DCC Linear Congruential Graphs: A New Class of Interconnection Networks

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Abstract—Let n be an integer and $F = \{f_i : 1 \le i \le t$ for some integer $t\}$ be a finite set of linear functions. We define a linear congruential graph $G(F, n)$ as a graph on the vertex set $V = \{0, 1, \ldots, n-1\}$, in which any $x \in V$ is adjacent to $f_i(x)$ mod n, $1 \le i \le t$. For a linear function g, and a subset V_1 of V we define a linear congruential graph $G(F, n, g, V_1)$ as a graph on vertex set V, in which any $x \in V$ is adjacent to $f_i(x)$ mod n, $1 \le i \le t$, and any $x \in V_1$ is also adjacent to $g(x)$ mod n.

These graphs generalize several well known families of graphs, e.g., the de Bruijn graphs. We give a family of linear functions, called DCC linear functions, that generate regular, highly connected graphs which are of substantially larger order than de Bruijn graphs of the same degree and diameter. Some theoretical and empirical properties of these graphs are given and their structural properties are studied.

Index Terms—Graph theory, interconnectons networks, network design, parallel processing, computer networks.

1 Introduction

In the design of massively parallel computers, one of the most important problems is the design of the interconnection network connecting the processors of the parallel computer. As stated in Hillis [18], the topology of the interconnection network imposes many performance restrictions. Any interconnection network can be represented as a graph in which the vertices correspond to the processors and the edges to the communication links. Thus, some of the properties of interconnection networks have been investigated in a graph-theoretical setting. For graph theoretical notation and terminology we follow [10]. The number of vertices of a graph will be called *the order* of the graph.

A family of graphs $\mathcal{F}$ that is suitable for interconnection networks should contain infinitely many graphs of different orders and degrees. Furthermore, graphs in $\mathcal{F}$ should be regular and of small degree, of relatively small diameter, high connectivity, and extensible, i.e., it is possible to construct a large graph in $\mathcal{F}$ from smaller graphs in the family. See, for example, [8], [18], [26] for more detailed discussions of these issues.

The problem of constructing large regular graphs of given degree d and diameter D, called (d, D) graph problem, and the related problem of constructing a network of given order n and degree d with smallest possible diameter, proposed first in [16], has attracted the attention of many researchers, see [3], [12] for surveys. The upper bound on the order n of a graph of degree $d > 2$ and diameter D, called the Moore bound, is as follows:

$$n \le (d(d-1)^D - 2)/(d-2).$$

For $D > 2$, and $d > 2$, the Moore bound cannot be attained [12]. Hence the main interest has been in constructing networks of degree $d > 2$ and diameter $D > 2$ whose order approaches Moore bound. See the tables in [12], [13] for the largest known graph orders for small values of d and D. These largest known graphs are constructed by different methods for different degrees and diameters, and they do not necessarily have many of the properties required for interconnection networks. The best families of graphs that contain large graphs of low diameter and have good network properties are de Bruijn graphs [11], and some of their variations, such as Kautz graphs [21], generalized de Bruijn graphs [14], [15] and Imase-Itoh graphs [19]. De Bruijn graphs were investigated for their suitability for communication networks, initially by Pradhan and Samanthan in [27], [28], Bermond and Peyrat in [8], and in many other papers by now. It is shown in [27], [28] that a binary de Bruijn network can solve a wide variety of classes of problems. Graphs having more vertices than de Bruijn graphs of the same degree and diameter can be obtained by a subgraph substitution into de Bruijn graphs [20].

Akers and Krishnamurthy [1] developed a formal group-theoretic model, called the Cayley graph model, for designing symmetric interconnection networks. Given a set of generators for a finite group G, the Cayley graph of G is the graph in which the vertices correspond to the elements of the group, and the edges correspond to the action of the generators. It is shown in [1] that two classes of these networks, called the star graphs and the pancake graphs satisfy many of the desirable network properties given above, and they can accommodate many more vertices than a hypercube of the same diameter and degree.

The order of graphs, as a function of the degree and the diameter, in the families of graphs mentioned above is poor

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compared to the Moore bound and to the bound obtained from the studies of random graphs [9]. Hence further improvements are needed.

Taking an approach which, in spirit, follows the same thrust as Akers and Krishnamurthy in [1], Opatrny and Sotteau proposed in [25] *Linear congruential graphs*. In a linear congruential graph of order n, the vertices are integers between 0 and $n-1$ and the adjacencies are defined by a set of linear functions. These graphs generalize de Bruijn graphs and some other families of graphs.

In this paper we define a new subfamily of linear congruential graphs, called *DCC linear congruential graphs* (DCC for Disjoint Consecutive Cycles), and investigate in detail several properties of graphs in this class. DCC linear congruential graphs are highly connected, regular graphs which can be constructed for any fixed degree d and any order n such that n contains a multiple factor. These graphs are much larger than de Bruijn graphs of the same degree and diameter.

In Section 2 of this paper we give a definition of linear congruential graphs, and some sufficient conditions for linear congruential graphs to be regular. Furthermore, the class of DCC linear congruential graphs is introduced.

In Section 3 we discuss DCC linear congruential graphs of order 2^p for some integer p. It is shown that DCC linear congruential graphs of even degree are maximally connected. Some structural properties of these graphs are discussed. An upper bound on the diameter of DCC linear congruential graphs of degree 4 in terms of their order is given. However, this bound is not close to the actual value of the diameters of generated graphs. We give tables of diameters of DCC graphs of various orders and degrees and give some conjectures concerning the diameter of DCC linear congruential graphs.

Since DCC graphs satisfy many of the requirements of interconnection networks stated above, they could be considered as an alternative for very large interconnection networks.

2 Linear Congruential Graphs

We use N to denote the set of nonnegative integers.

Definition 2.1. *Let n be a positive integer and F be a finite set of t linear functions for some integer t, $F = \{f_i(x) = (a_i x + c_i) : 1 \le i \le t$, where $a_i, c_i \in N\}$. We define a linear congruential graph $G(F, n)$ as the graph on the vertex set $V = \{0, 1, \ldots, n-1\}$, in which any $x \in V$ is adjacent to $f_i(x)$ mod n, for every i, $1 \le i \le t$. For a subset V_1 of V and a linear function g, we define a linear congruential graph $G(F, n, g, V_1)$ as the graph on vertex set V, in which any $x \in V$ is adjacent to $f_i(x)$ mod n, for every i, $1 \le i \le t$, and any $x \in V_1$ is also adjacent to $g(x)$ mod n.*

We call the functions in F and $F \cup \{g\}$, the *generators* of $G(F, n)$ and $G(F, n, g, V_1)$, respectively. For any linear function f we call the graph $G(\{f\}, n)$ *the graph generated by f on* $\{0, 1, \ldots, n-1\}$. See Fig. 1 for an example of a linear congruential graph.

We will show that, for a suitable chosen set of generators F, the graph $G(F, n)$ is a regular graph of even degree. If n is even then for a suitable chosen set of generators $F \cup \{g\}$, the graph $G(F, n, g, V_1)$ will be shown to be a regular graph of odd degree.

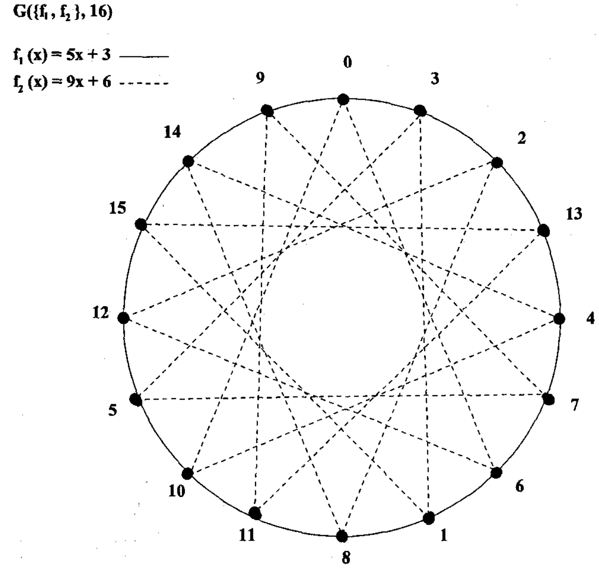

Fig. 1. A linear congruential graph.

Clearly, $G(F, n)$, (or $G(F, n, g, V_1)$), could be also considered as a directed graph in which there is an arc from any $x \in V$ to $f_i(x)$ *mod n*, $1 \le i \le t$, (or from any $x \in V$ to $f_i(x)$ *mod n*, $1 \le i \le t$, and from any vertex x of V_1 to $g(x)$). In this paper we restrict our attention to undirected graphs.

Graphs with edge sets defined by linear functions modulo the order of the graph were investigated in [22] as possible expanders but no other of their properties was considered there.

The class of linear congruential graphs is a very broad family of graphs. By imposing some restrictions on the values of the constants of the generators, we can obtain subfamilies of linear congruential graphs. For example, if in a graph $G(F, n)$ all multiplicative constants of the generators in F are equal to 1, and if one of the generators is the function $x + 1$, then we obtain a distributed loop graph [4]. The de Bruijn graph $G(d, D)$ of degree d and diameter D is isomorphic to a linear congruential graph $G(F, d^D)$ where $F = \{dx + i : 0 \le i \le d-1\}$. Similarly Kautz graphs and generalized de Bruijn graphs (see [2], [3], [21], [15]) can all be obtained as subfamilies of linear congruential graphs.

Our goal is to define a new subfamily of linear congruential graphs that would have very good network properties. We first study the structure of linear congruential graphs generated by single generators. The results of this study will be used to obtain sufficient conditions for a set of generators to generate regular graphs.

Let n be a positive integer, $f(x) = ax + c$ be a linear function, where $a, c \in N$, and let $x_0 \in \{0, 1, \ldots, n-1\}$. The sequence of integers $x_0, x_1, \ldots, x_i, \ldots$, defined by $x_i = f(x_{i-1})$ mod n for $i > 0$, called a linear congruential sequence [23], is a periodic sequence with period length less than or equal to n. By the definition of the edge set of the linear congruential graphs, the elements of the sequence corresponding to a period of the linear congruential sequence form a cycle in the graph generated by f on $\{0, 1, \ldots, n-1\}$. We now state two lemmas that will be needed in our paper. The first lemma gives the necessary and sufficient conditions on a linear function f and an integer n to define a linear congruential sequence of period length equal to n. Its proof is given in detail in [23].

LEMMA 2.2. [23] *Let $f(x) = ax + c$ be a linear function, n be a positive integer, and $x \in \{0, 1, 2, \ldots, (n-1)\}$. For a given x_0, the linear congruential sequence $x_0, x_1, x_2, \ldots, x_i, \ldots$, defined for $i \geq 1$ by $x_i = (ax_{i-1} + c) \bmod n$, has a period of length n if and only if*

1) *$gcd(c, n) = 1$,*
2) *$b = (a-1)$ is a multiple of every prime factor of n; b is also a multiple of 4, if n is a multiple of 4.*

If a linear function f satisfies the conditions of Lemma 2.2 with respect to n then the graph $G(\{f\}, n)$ consists of a Hamiltonian cycle on $\{0, 1, \ldots, n-1\}$. This is the case, for example, when n is a power of 2 and f is a function from the set $\{5x + 1, 5x + 3, \ldots, 9x + 1, 9x + 3, \ldots, 17x + 1, \ldots\}$.

For a graph $G(F, n)$, an interesting case arises when each linear function in F satisfies the conditions of Lemma 2.2 with respect to n, and $G(F, n)$ thus consists of $|F|$ Hamiltonian cycles (not necessarily edge disjoint without additional conditions on the constants in the functions). Linear congruential sequences with maximum period lengths are used in pseudorandom number generators. Since random graphs are almost always of low diameter [9], it was expected that $G(F, n)$ graphs consisting of edge disjoint Hamiltonian cycles corresponding to linear congruential sequences of period length n could be graphs of low diameter. Such graphs were considered in [25], where many graphs whose orders are much larger than the orders of de Bruijn graphs of the same degree and diameter were constructed. One difficulty encountered in that case is the choice of the constants of the linear functions in F so that any two functions in F define edge-disjoint Hamiltonian cycles, which would produce a regular graph. Our subsequent investigations indicated that graphs with smaller diameter can be obtained when only one of the functions in F generates a Hamiltonian cycle, and all other functions generate several cycles on $\{0, 1, \ldots, n-1\}$. This is the case considered in this paper.

Lemma 2.3 gives sufficient conditions on a linear function to generate several cycles of equal length on the vertex set $\{0, 1, \ldots, n-1\}$.

LEMMA. 2.3. *Let n be a positive integer that contains at least one multiple factor, i.e., $n = k^p m$ for some integers $k > 1$, $p \geq 2$, and m. Let c be an integer such that $gcd(c, n) = 1$. Let b be a multiple of every prime factor of n; b is also a multiple of 4, if n is a multiple of 4. For any i, $1 \leq i \leq p + 1$, let $f_i(x) = (k^{i-1} b + 1) x + k^{i-1} c$.*

Then, the function f_i generates k^{i-1} vertex-disjoint cycles of length $\frac{n}{k^{i-1}}$ on the set $\{0, 1, \ldots, n-1\}$. The vertex sets of these cycles are the sets

$$A_{1i} = \{0, k^{i-1}, 2k^{i-1}, \ldots, n-k^{i-1}\},$$

$$A_{2i} = \{1, k^{i-1}+1, \ldots, n-k^{i-1}+1\}, \ldots,$$

$$A_{k^{i-1}i} = \{k^{i-1}-1, 2k^{i-1}-1, \ldots, n-1\}.$$

Furthermore, there is an edge between two vertices x and y in the graph generated by f_i only if $|y-x|$ is divisible by k^{i-1} but not by k^i.

PROOF. Let i be an integer, $1 \leq i \leq p+1$, and r be an integer, $0 \leq r \leq k^{i-1} - 1$. Consider any integer $k^{i-1}q + r$ of the set $A_{(r+1)i}$. Then

$$\begin{aligned} f_i(k^{i-1}q + r) \bmod n &= ((k^{i-1}b+1)(k^{i-1}q+r) + k^{i-1}c) \bmod k^p m \\ &= (k^{i-1}(k^{i-1}b+1)q + k^{i-1}(br+c)) \bmod k^p m + r \\ &= k^{i-1}[((k^{i-1}b+1)q + (br+c)) \bmod k^{p-i+1} m] + r. \end{aligned}$$

Thus $f_i(k^{i-1}q + r) \bmod n$ also belongs to the same set $A_{(r+1)i}$.

Now, let us consider the linear function $h(x) = (k^{i-1}b + 1)x + (br + c)$. Since b is divisible by any prime factor of n, and $gcd(c, n) = 1$, then $gcd(br + c, k^{p-i+1}m) = 1$ and so $k^{i-1}b + 1$ satisfies condition 1) of Lemma 2.2 with respect to $k^{p-i+1}m$. Also $h(x)$ satisfies condition 2) of Lemma 2.2 with respect to $k^{p-i+1}m$. Hence, by Lemma 2.2, the linear congruence $h(x) \bmod k^{p-i+1}m$ generates a linear congruential sequence of period $k^{p-i+1}m$ on the set $\{0, 1, \ldots, k^{p-i+1}m - 1\}$.

This implies that, for every r, $0 \leq r \leq k^{i-1} - 1$, the function f_i generates a cycle of length $k^{p-i+1}m$ on the set

$$A_{(r+1)i} = \{0 + r, k^{i-1} + r, \ldots, n - k^{i-1} + r\}.$$

If (x, y) is an edge in the graph generated by f_i, then we have

$$\begin{aligned} |y - x| &= |f_i(x) \bmod n - x| \\ &= |(k^{i-1}b+1)x + k^{i-1}c - sn - x| \\ &= |k^{i-1}bx + k^{i-1}c - sn| \\ &= |k^{i-1}(bx + c - sk^{p-i+1}m)| \end{aligned}$$

for some integer s. Since b is divisible by k and c is not divisible by k, $|y-x|$ is divisible by k^{i-1} but not by k^i. □

A linear function defined as in Lemma 2.3, which generates k^j disjoint cycles, will be called of *cycle type* k^j on the set $\{0, 1, \ldots, n-1\}$. Similarly a function satisfying Lemma 2.2 which generates a Hamiltonian cycle will be called of *cycle type* 1 on the set $\{0, 1, \ldots, n-1\}$.

For example, function $9x + 2$ is of cycle type 2 and $17x + 4$ is of cycle type 4 on the set $\{0, 1, \ldots, 2^p\}$ for $p \geq 2$.

In the lemma above, if m is equal to 1 or 2 and i is equal to $p + 1$, or if k is equal to 2, m is equal to 1, and i is equal to p, then the generator f_i generates loops, or multiple edges in the graph. Since we are interested in simple graphs without loops and multiple edges, in our constructions we only use generators that give cycles of lengths larger than 2. We can achieve it by stipulating that any generator generates at most k^j cycles with $k^j < \frac{n}{2}$.

THEOREM. 2.4. *Let n be a positive integer that contains at least one multiple factor, i.e., $n = k^p m$ for some integers $k > 1$, $p \geq 2$,* *and m. For an integer t in* $\{0, 1, \cdots p + 1\}$, *let F be a set of t linear functions such that:*

1) *each function in F is of cycle type* k^j *on* $\{0, 1, \ldots, n-1\}$ *for some* j, $0 \le j \le p$, *such that* $k^j < \frac{n}{2}$.
2) *there is exactly one function in F of cycle type 1 on the set* $\{0, 1, \ldots, n-1\}$,
3) *any two functions in F are of different cycle types on* $\{0, 1, \ldots, n-1\}$.

Then the graph $G(F, n)$ *is a regular, connected graph of degree* $2t$.

If $k = 2$, *and for some linear function g the set* $F \cup \{g\}$ *satisfies the three conditions above, and g is of cycle type* 2^l *on* $\{0, 1, \ldots, n-1\}$ *for some l,* $l \ge 1$, *then the graph* $G(F, n, g, V_1)$, *where* $V_1 = \{0, 1, \ldots, 2^l - 1, 2^{l+1}, 2^{l+1} + 1, \ldots, 2^{l+2} - 1, \ldots\}$, *is a regular, connected graph of degree* $2t + 1$.

PROOF. Since F includes a function that generates a Hamiltonian cycle, the graphs $G(F, n)$ and $G(F, n, g, V_1)$ are connected. Since any two functions in F or $F \cup \{g\}$ are of different cycle types with respect to n, the cycles generated by these two functions are edge disjoint by Lemma 2.3. Thus each function in F contributes 2 to the degree of each vertex and the graph $G(F, n)$ is of degree $2t$.

We have chosen V_1 so that the graph $G(F, n, g, V_1)$ contains every second edge of the cycles generated by g on the set$\{0, 1, \ldots, n-1\}$. Since $k = 2$, each cycle is of even length by Lemma 2.3 and thus g restricted to V_1 generates a perfect matching in the graph. Therefore, $G(F, n, g, V_1)$ is a regular graph of degree $2t + 1$. □

In our experiments the best results, with respect to the diameter of linear congruential graphs of order $n = k^p m$ and degree $2t$ or $2t + 1$, have been obtained when the generators satisfy the conditions of Theorem 2.4 and, furthermore, when the set F or $F \cup \{g\}$ contains a function of cycle typek^j for each j, $0 \le j \le t-1$ or j, $0 \le j \le t$. With this in view we introduce the notion of a *Disjoint Consecutive Cycles set of generators.*

DEFINITION 2.5. *Let n be a positive integer that contains at least one multiple factor, i.e.,* $n = k^p m$ *for some integers* $k > 1$, $p \ge 2$, *and m. For any given* $t \in \{1, \cdots, p + 1\}$, *such that* $k^{t-1} < \frac{n}{2}$, *we say that a set F of t linear functions is a Disjoint Consecutive Cycles set (DCC set for short) with respect to the integer n if for each j,* $0 \le j \le t - 1$, *there is exactly one function in F of cycle type* k^j *on the set* $\{0, 1, \ldots, n-1\}$.

For example, $\{5x + 1, 9x + 2, 17x + 4\}$ and $\{5x + 3, 9x + 10, 17x + 12\}$ are DCC sets of generators, each of cycle type 1, 2, and 4 with respect to 2^p for $p \ge 4$, while $\{4x + 1, 10x + 3, 28x + 9\}$ is a DCC set of generators of cycle types 1, 3, and 9 with respect to 3^p for $p \ge 3$.

DEFINITION 2.6. *Let n be a positive integer that contains at least one multiple factor, i.e.,* $n = k^p m$ *for some integers* $k > 1$, $p \ge 2$, *and m. For any given* $t \in \{1, \cdots, p + 1\}$ *such that* $k^{t-1} < \frac{n}{2}$, *we define a* DCC linear congruential graph $G_{2t}(F, n)$ *as a linear congruential graph* $G(F, n)$ *generated by a DCC set F of t linear functions with respect to n. If* $k = 2$, $t \le p$, *and g is a linear function of cycle type* 2^t, *we define a* DCC linear congruential graph $G_{2t+1}(F, n, g)$ *as a linear congruential graph* $G(F, n, g, V_1)$ *where*

$$V_1 = \{0, 1, \ldots, 2^t - 1, 2^{t+1}, 2^{t+1} + 1, \ldots, 2^{t+2} - 1, \ldots\}.$$

Notice that the graph in Fig. 1 is a DCC linear congruential graph. Theorem 2.4 gives immediately the following result.

COROLLARY 2.7. *Let n be a positive integer that contains at least one multiple factor, i.e.,* $n = k^p m$ *for some integers* $k > 1$, $p \ge 2$, *and m. For any given* $t \in \{1, \cdots, p + 1\}$, *such that* $k^{t-1} < \frac{n}{2}$, *let F be a DCC set of t linear functions. Then the graph* $G_{2t}(F, n)$ *is a regular, connected graph of degree 2t. If* $k = 2$ *and* $t \le p$ *then the graph* $G_{2t+1}(F, n, g)$ *is a regular, connected graph of degree* $2t + 1$.

A question could be asked whether our choice of set V_1 in defining a DCC linear congruential graph of odd degree was good. Clearly, if the function g of cycle type 2^t generates a perfect matching on the set $\{0, 1, \ldots, n-1\}$ when the function is applied to the set V_1, the function g would also generate a perfect matching when applied to the set

$$V_2 = \{0, 1, \ldots, n-1\} - V_1,$$

which could possibly produce better results. The following theorem disproves this possibility.

THEOREM. 2.8. *Let F and* $F \cup \{g\}$ *be DCC sets of linear functions with respect to* $n = 2^p m$ *for some integer* $p > 1$, *and* $|F| = t$. *Let* $V_1 = \{0, 1, \ldots, 2^t - 1, 2^{t+1}, 2^{t+1} + 1, \ldots, 2^{t+2} - 1, \ldots\}$, *and* $V_2 = \{0, 1, \ldots, n-1\} - V_1$. *There exists a linear function h such that the graph* $G_1 = G(F, n, g, V_2)$ *is identical to the graph* $G_2 = G(F, n, h, V_1)$.

PROOF. For any element $x \in V_2$, $g(x) \bmod n$ is an element of the set V_1. Since g is a one-to-one linear mapping of $\{0, 1, \ldots, n-1\}$ onto $\{0, 1, \ldots, n-1\}$, there exists a linear function h which is an inverse of g on the set $\{0, 1, \ldots, n-1\}$. Therefore, the matching defined by h on V_1 is identical to the matching defined by g on V_2, and h is of the same cycle type as g. □

3 PROPERTIES OF DCC LINEAR CONGRUENTIAL GRAPHS

In this section we will study in detail the DCC linear congruential graphs whose order n is a power of 2, i.e., $n = 2^p$ for some integer p. Thus, the generators considered in this section will be a DCC set of linear functions

$$F = \{f_i : 1 \le i \le t\}$$

for some $t \le p - 2$, with

$$f_i \in \{a_i x + c_i : a_i = 2^{i+1} b' + 1,\ c_i = 2^{i-1}(2r + 1)\ |b',\ r \in N\}.$$

These functions satisfy the conditions of Lemma 2.3 for $i > 1$, and the conditions of Lemma 2.2 for $i = 1$ with respect to the powers of 2. Recall that, for any $i \in \{1, \cdots, p - 1\}$, the function f_i is of cycle type 2^{i-1} on the set $\{0, 1, \ldots, 2^p - 1\}$. Thus the 2^{i-1} disjoint cycles generated by f_i partition the vertex set $\{0, 1, \ldots, 2^p - 1\}$ into 2^{i-1} disjoint subsets of cardinality 2^{p-i+1},

$$A_{1i} = \{0, 2^{i-1}, 2^i, \ldots, n - 2^{i-1}\},$$
$$A_{2i} = \{1, 2^{i-1} + 1, 2^i + 1, \ldots, n - 2^{i-1} + 1\}, \ldots, +1\}, \ldots,$$
$$A_{2^{i-1}i} = \{2^{i-1} - 1, 2^i - 1, \ldots, n - 1\}.$$

In particular, f_2 generates 2 cycles of length $\frac{n}{2}$, one on the set of even numbered vertices, the other one on the set of odd numbered vertices. Furthermore, the partition defined by f_i is a refinement of the partition defined by f_{i-1} for $i \geq 2$. More precisely, we can state the following result, that will be used in the proofs of the properties of the graphs $G_{2t}(F, 2^p)$ and $G_{2t+1}(F, 2^p, g)$.

PROPOSITION 3.1. *Let $G = G_{2t}(F, 2^p)$ be a DCC linear congruential graph on the vertex set $\{0, 1, \cdots, 2^p - 1\}$, where $F = \{f_i : 1 \leq i \leq t\}$, with $f_i(x) = a_i x + c_i$ verifying the conditions given above. Then, the subgraphs G_1 and G_2, induced by the vertex sets $V_1 = \{0, 2, \cdots, 2^p - 2\}$ and $V_2 = \{1, 2, \cdots, 2^p - 1\}$, are isomorphic to the DCC linear congruential graphs $G_{2t-2}(F', 2^{p-1})$ and $G_{2t-2}(F'', 2^{p-1})$ where*

$$F' = \{f_i' : 1 \leq i \leq t-1\}$$
$$F'' = \{f_i'' : 1 \leq i \leq t-1\}$$

with

$$f_i'(x) = a_{i+1}x + \frac{c_{i+1}}{2} \text{ and}$$
$$f_i''(x) = a_{i+1}x + \frac{c_{i+1} + a_{i+1} - 1}{2}$$

respectively. Moreover, G is the edge-disjoint union of G_1, G_2, and of the Hamiltonian cycle induced by f_1 which forms a bipartite graph of degree 2 on (V_1, V_2).

PROOF. For any i, $1 \leq i \leq t-1$, the constant a_{i+1} of functions f_i' and f_i'' is equal to $2^{i+1}b'' + 1$ for some integer b'', and the constants $c_{i+1}/2$ and $(c_{i+1} + a_{i+1} - 1)/2$ are divisible by 2^{i-1} but not by 2^i. Thus f_i', f_i'' is of cycle type 2^{i-1} for $1 \leq i \leq t-1$, and the graphs $G_{2t-2}(F', 2^{p-1})$ and $G_{2t-2}(F'', 2^{p-1})$ are DCC linear congruential graphs.

We will show that the functions $h_1(x) = x/2$ and $h_2(x) = (x-1)/2$ define an isomorphism between G_1 and $G_{2t-2}(F', 2^{p-1})$, G_2 and $G_{2t-2}(F'', 2^{p-1})$, respectively. Consider first the graph G_1. Let i be an integer, $2 \leq i \leq t$, and x be a vertex in V_1. Since any vertex in V_1 is an even integer, $x = 2j$ for some j. Thus,

$$f_i(x) = f_i(2j) = a_i 2j + c_i = 2(a_i j + c_i / 2) = 2f_{i-1}'(j).$$

If $(x, f_i(x))$ is an edge in G_1 then $(h_1(x), h_1(f_i(x))) = (j, f_i(2j)/2) = (j, f_{i-1}'(j))$ is an edge in $G_{2t-2}(F', 2^{p-1})$. Similarly, for every y in $\{0, 1, \ldots, 2^{p-1}\}$, if $(y, f_{i-1}'(y))$ is an edge in $G_{2t-2}(F', 2^{p-1})$ then $(h_1^{-1}(y), h_1^{-1}(f_{i-1}'(y))) = (2y, f_i(2y))$ is an edge in G_1. Thus, h_1 defines an isomorphism between G_1 and $G_{2t-2}(F', 2^{p-1})$.

Consider now the graph G_2. Let i be an integer, $2 \leq i \leq t$, and x be a vertex in V_2. Since any vertex in V_2 is an odd integer, $x = 2j + 1$ for some j. Thus,

$$\begin{aligned} f_i(x) &= f_i(2j+1) = a_i 2j + a_i + c_i \\ &= 2(a_i j + (c_i + a_i - 1)/2) + 1 \\ &= 2f_{i-1}''(j) + 1 \end{aligned}$$

If $(x, f_i(x))$ is an edge in G_2 then

$$\begin{aligned} (h_2(x), h_2(f_i(x))) &= (j, (f_i(2j+1) - 1)/2 \\ &= (j, f_{i-1}''(j)) \end{aligned}$$

is an edge in $G_{2t-2}(F'', 2^{p-1})$. Similarly, for every y in $\{0, 1, \ldots, 2^{p-1}\}$, if $(y, f_{i-1}''(y))$ is an edge in $G_{2t-2}(F'', 2^{p-1})$ then

$$\begin{aligned} (h_2^{-1}(y), h_2^{-1}(f_{i-1}''(y))) &= (2y+1, 2f_{i-1}''(y)+1) \\ &= (2y+1, f_i(2y+1)) \end{aligned}$$

is an edge in G_2. Thus, h_2 is an isomorphism between G_2 and $G_{2t-2}(F'', 2^{p-1})$.

The fact that G is the edge-disjoint union of G_1, G_2, and of the Hamiltonian cycle induced by f_1 which forms a bipartite graph of degree 2 on (V_1, V_2) is obvious. □

We first consider the vertex-connectivity (connectivity for short) of the DCC linear congruential graphs.

THEOREM 3.2. *Let $t \leq p$ be an integer, and $F = \{f_i : 0 \leq i \leq t-1\}$ be defined as above. Then the DCC linear congruential graph $G_{2t}(F, 2^p)$ is $2t$-connected.*

PROOF. The proof is by induction on t. The result is obviously true for $t = 1$ since in this case the DCC linear congruential graph is a cycle of length 2^p and, therefore, is 2-connected. Assume that, for any appropriate DCC set F' of $t-1$ linear functions, the graph $G_{2(t-1)}(F', 2^{p-1})$ is $2(t-1)$-connected. Consider the partition of the set of vertices of a DCC linear congruential graph $G_{2t}(F, 2^p)$ into the vertex sets V_1, V_2 of even, odd numbered vertices, on which f_2 generates cycles of lengths 2^{p-1}, say C_1 and C_2. By Proposition 3.1, the subgraphs G_1 and G_2 induced by V_1 and V_2 are isomorphic to DCC linear congruential graphs $G_{2(t-1)}(F', 2^{p-1})$ and $G_{2(t-1)}(F'', 2^{p-1})$, respectively which, by the induction hypothesis, are $2(t-1)$-connected. Consider any two vertices x and y of $G_{2t}(F, 2^p)$.

Case 1. Vertices x and y are both in V_1 (see Fig. 2a). Since by the induction hypothesis G_1 is $2(t-1)$-connected, there exist $2t-2$ vertex-disjoint paths between x and y in G_1. It is easy to exhibit two more vertex disjoint paths in

$G_{2t}(F, 2^p)$ depending on the relative position of the vertices $f_1(x), f_1(y), f_1^{-1}(x), f_1^{-1}(y)$ on the cycle C_2 as follows. If both $f_1^{-1}(x)$ and $f_1^{-1}(y)$ are on the same part of the cycle C_2 between $f_1(x)$ and $f_1(y)$, then the paths are:

Path 1: x, $f_1^{-1}(x)$, part of C_2 to $f_1^{-1}(y)$, y

Path 2: x, part of C_2 not containing $f_1^{-1}(x)$ to $f_1(y)$, y;

otherwise

Path 1: $x, f_1(x)$, part of C_2 containing $f_1^{-1}(x)$ to $f_1(y)$, y

Path 2: $x, f_1(x)$, part of C_2 containing $f_1^{-1}(y)$ to $f_1(y)$, y

Case 2. Vertex x is on C_1 and y is on C_2. Assume first that $y \notin \{f_1(x), f_1^{-1}(x)\}$ (see Fig. 2b). Let S_1 be a subset of $2t - 2$ vertices of $V_1 - \{x, f^{-2}(x)\}$ containing $f_1(y)$ and $f_1^{-1}(y)$. Denote the vertices of S_1 different from $f_1(y)$ and $f_1^{-1}(y)$ as $a_1, a_2, \cdots, a_{2t-4}$. By the induction hypothesis, G_1 is $2(t-1)$-connected, and thus, by Menger's theorem, there exists in G_1, a set $\mathcal{P}_1$ of $2t - 2$ vertex disjoint paths between x and the vertices of S_2. By Proposition 3.1, f_1 defines a matching of edges between these vertices and $2t - 4$ vertices of V_2. Let S_2 be the subset of V_2 containing $f_1(a_1), \cdots, f_{2t-4}(a_{2t-4})$ and the two vertices $f_1(x)$ and $f_1^{-1}(x)$ which are necessarily different from the previous ones by the choice of S_1. Again, by induction hypothesis G_2 is $2(t-1)$-connected, and thus by Menger's theorem, there exists in G_2 a set $\mathcal{P}_2$ of $2t - 2$ vertex disjoint paths between the vertices of S_2 and y. We are now able to exhibit $2t$ vertex disjoint paths between x and y in $G_{2t}(F, 2^p)$. For every i, $1 \le 1 \le 2t - 4$, take the following path:

x, path in $\mathcal{P}_1$ between x and $a_i, f_1(a_i)$, path in $\mathcal{P}_2$ between $f_1(a_i)$ and y.

The last four paths are defined as follows.

Path 1: $x, f_1(x)$, path between $f_1(x)$ and y in G_2;

Path 2: x, $f_1^{-1}(x)$, path between $f_1^{-1}(x)$ and y in G_2;

Path 3: x, path in G_1 between x and $f_1^{-1}(y)$, y;

Path 4: x, path in G_1 between x and $f_1(y)$, y.

The proof is very similar if $y = f_1(x)$ (or $y = f_1^{-1}(x)$, respectively). In that case paths 1 and 3 (or paths 2 and 4, respectively) above are reduced to the edge xy. It is possible to add one more vertex a_{2t-3} in the set S_1 so that it is still of cardinality $2t - 2$ and it avoids $f_1^{-1}(y)$ (or $f_1(y)$, respectively) which is equal to x. Thus, following the same reasoning as above, where $2t - 4$ is replaced by $2t - 3$, we still have $2t$ vertex disjoint paths between x and y. □

NOTE 3.3. The graph $G_{2t+1}(F, 2^p, g)$ is at least 2t-connected since it contains $G_{2t}(F, 2^p)$ as a subgraph. However, there are cases when $G_{2t+1}(F, 2^p, g)$ is not $2t + 1$ connected, e.g., $G_3(\{5x + 3\}, 8, 9x + 2)$ is not 3-connected.

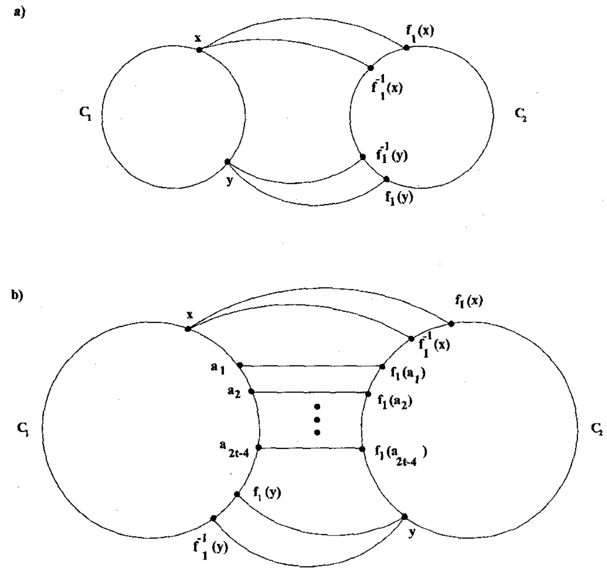

Fig. 2. (a) Case 1 in proof of Theorem 3.2.; (b) Case 2 in proof of Theorem 3.2.

The problem of deriving the value of the diameter of $G_k(F, 2^p)$ seems to be difficult. This should not be surprising since the problem of determining the precise value of the diameter is difficult even for some simpler graphs, as for example the multiple loops graphs [4]. Below we give an upper bound on the diameter of $G_4(F, 2^p)$ in case when the function f_2 is equal to $x + 2$. Notice that this function f_2 is of cycle type 2 since $(x + 2) \bmod 2^p = ((2^p + 1)x + 2) \bmod 2^p$.

THEOREM 3.4. *Let $F = \{ax + c, x + 2\}$, where a and c satisfy the conditions of Lemma 2.2, and let p be an integer, $p \ge 3$. Then the diameter of the DCC linear congruential graph $G_4(F, 2^p)$ is smaller than or equal to $\lceil \log_a 2^{p-1} \rceil (a+1)/2$.*

PROOF. Let x be a vertex of the graph. Clearly, $f_2(x) = x + 2$, and $f_2^{-1}(x) = x - 2$. Consider the value of

$$z = g_s^{j_s}(f_1(g_{s-1}^{j_{s-1}}(f_1(\cdots(g_1^{j_1}(x))))))$$

where $0 \le j_l \le (a-1)/2$ and g_l is either equal to f_2, or to f_2^{-1} for every l, $1 \le l \le s$. Notice that for any value of z above, there is a path from x to $z \bmod 2^p$ of length $j_1 + j_2 + \cdots j_s + s - 1$ in $G_4(F, 2^p)$. Since a and c are both odd, the parity of $f_1(x)$ is different from the one of x.

If $s = 1$ then $z = g_1^{j_1}(x)$, and the possible values of z are a integers of the same parity as x in the set $R_1 = \{x - (a-1), x - (a-1) + 2, \ldots, x + (a-1)\}$. Consider now the possible values of z for $s = 2$, i.e., $z = g_2^{j_2}(f_i(g_1^{j_1}(x)))$. First, by applying the function f_1 to the elements of R_1, we obtain the set

$$\begin{aligned} X_1 &= \{f_1(x-(a-1)), f_1(x-(a-1)+2), \ldots, f_1(x+(a-1))\} \\ &= \{f_1(x)-(a^2-a), f_1(x)-(a^2-a)+2a, \ldots, f_1(x)+(a^2+a)\}. \end{aligned}$$

All values in X_1 are of the same parity as $f_1(x)$ and the difference between any two consecutive values in the set is equal to $2a$. Now, by applying the function $g_2^{j_2}$ to the values in the set X_1 for all values of j_2, $0 \le j_2 \le (a-1)/2$ and g_2 being either f_2, or f_2^{-1}, we obtain all values in the set

$$\begin{aligned} R_2 &= \{f_1(x) - (a^2 - a) - (a-1),\ f_1(x) - (a^2 - a) \\ &\quad - (a-1) + 2,\ \ldots,\ f_1(x) + (a^2 - a) + (a-1)\} \\ &= \{f_1(x) - a^2 + 1,\ f_1(x) - a^2 + 3,\ \ldots,\ f_1(x) + a^2 - 1\} \end{aligned}$$

This set contains a^2 elements, all of them of the same parity. Similarly, for any value of s, the possible values of z are a^s integers in the set

$$\{f_1^{s-1}(x) - (a^s - 1),\ f_1^{s-1}(x) - (a^s - 1) + 2,\ \ldots,\ (f_1^{s-1}(x) + (a^s - 1)\}$$

having the same parity as $f_1^{s-1}(x)$. Thus, if we choose s to be the smallest integer such that $a^s \ge 2^{p-1}$, i.e., $s = \lceil \log_a 2^{p-1} \rceil$, then from x there is a path to any vertex of the same parity as $f_1^{s-1}(x)$ whose length is bounded by $s(a-1)/2 + s - 1 = \lceil \log_a 2^{p-1} \rceil (a+1)/2 - 1$. Since f_1 is a one-to-one mapping of all the vertices of the same parity of the graph onto the vertices of the opposite parity, by extending all these paths by an edge defined by f_1 we can reach the remaining vertices of the graph. We can conclude that there is a path from x to any other vertex of the graph of length not more than $\lceil \log_a 2^{p-1} \rceil (a+1)/2$. □

In Table 1, we give the values of the diameters of some of the graphs $G_{2t}(F, 2^p)$ and $G_{2t+1}(F, 2^p, g)$ which have a low diameter, for $9 \le p \le 17$ and degrees between 3 and 10. These values are in fact much smaller than the upper bound from the above theorem. Each entry in the table specifies the value of the diameter, and below the diameter we give the constants of the linear functions generating the graph. For graphs of order up to 2^{14} we have calculated the distances between all pairs of vertices of the graphs.

For graphs of larger order we have calculated the distances only for large segments of vertices.

In all the cases the DCC linear congruential graphs are much larger than the de Bruijn graphs of the same degree and diameter. For example, for degree 4 and diameter 10, there is a DCC linear congruential graph of order 8,192 while the de Bruijn graph is of order 1,024. Similarly, there is a DCC linear congruential graph of degree 9, and diameter 5 which has more vertices than the de Bruijn graph of degree 10, and diameter 5.

The diameter of DCC linear congruential graphs is sensitive to the choice of the constants of the generators. For example, the diameter of a DCC graph increases significantly when its set of generators contains a pair of functions $5x + 1$, $9x + 2$ which are commutative. As seen from the tables, we obtained the best results with respect to the diameter of DCC graphs when:

1) the multiplicative constants of the generators are all distinct,
2) the multiplicative constants increase with the cycle type of the generators, however the increases are as small as possible,
3) the generators are not commutative, i.e., the function $f_i \circ f_j$ is different from the function $f_j \circ f_i$ for $i \ne j$.

In general our experiments have indicated that all DCC linear congruential graphs of same degree and order obtained with the above restrictions on the generators have almost the same diameter.

Table 2 gives additional DCC linear congruential graphs, some of them of orders that are not powers of 2.

The results in Tables 1 and 2 lead us to propose the following conjecture:

CONJECTURE 3.5. *For the functions* $f_1 = 5x + 3$, $f_2 = 9x + 2$,

$$\textit{diameter}\ (G_3(\{f_1\}, 2^p, f_2)) \le \lceil 1.2\, p \rceil$$
$$\textit{diameter}\ (G_4(\{f_1, f_2\}, 2^p)) \le \lceil 0.8\, p \rceil.$$

Notice that the Moore bound implies that the diameter of graphs of order n and degree 3 is greater than or equal to $\log_2 n - 2/3$, and the diameter of graphs of order n and degree 4 is greater than or equal to $0.631 \log_2 n - 1/2$. The best general construction of graphs of order n and degree 3 or 4 was given by Jerrum and Skyum [20]. For degree 3, it gives graphs of diameter $1.47 \log_2 n + O(1)$, and for degree 4, the diameter is $0.9083 \log_2 n + O(1)$.

In Proposition 3.1, we showed that a DCC linear congruential graph $G_{2t}(F, 2^p)$ of degree $2t$ and order 2^p can be decomposed into two vertex disjoint DCC linear congruential graphs $G_{2t-2}(F', 2^{p-1})$ and $G_{2t-2}(F'', 2^{p-1})$ of degree $2t - 2$ and order 2^{p-1}. Furthermore, there is a bipartite graph of degree 2 on the partition defined by the two disjoint sets of vertices.

We now give a construction of a DCC linear congruential graph of order 2^{p+1} from two copies of DCC linear congruential graphs of order 2^p.

CONSTRUCTION 3.6. *A DCC linear congruential graph* $G_{2t}(F, 2^{p+1})$ *can be constructed from two copies of* $G_{2t}(F, 2^p)$ *as follows:*

1) *Denote the two copies of* $G_{2t}(F, 2^p)$ *as* H_1 *and* H_2. *Renumber the vertices of* H_2 *by adding* 2^p *to each vertex.*
2) *For every vertex* x *of* H_1, *if* (x, y) *is an edge in* H_1 *where* $y = f_j(x) \bmod 2^p$ *and* $2^p < f_j(x) \bmod 2^{p+1}$ *then remove the edge* (x, y) *in* H_1 *and the edge* $(x + 2^p, y + 2^p)$ *in* H_2 *and add instead edges* $(x, y + 2^p)$ *and* $(x + 2^p, y)$.

Notice that in the construction above, only a fraction of the edges is replaced since when the edge $(x, f_i(x) \bmod 2^p)$ is changed then the edge $(2x \bmod 2^p, f_j(2x) \bmod 2^p)$ is not changed.

Although the construction above is stated only for a DCC linear congruential graph of even degree, it is clear that the same construction can be also carried out for a DCC linear congruential graph of odd degree. See in Fig. 3 a construction of $G_3(\{f_1\}, 32, g)$ from $G_3(\{f_1\}, 16, g)$ for $f_1(x) = 5x + 3$, $g(x) = 9x + 2$. The edges that are to be replaced are drawn in thick lines.

TABLE 1
DIAMETERS OF DCC LINEAR CONGRUENTIAL GRAPHS $G_{2t}(F,\ 2^p)$, AND $G_{2t+1}(F,\ 2^p,\ g)$

n / deg	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}	2^{17}
3	10	12	13	14	15	17	18	19	20
	5,15 9,14	5,3 9,2	5,3 9,2	5,11 9,6	5,11 9,6	5,3 9,2	5,3 9,2	5,3 9,2	5,3 9,2
4	7	8	9		10	11	12		13
	5,7 9,2	5,7 9,2	5,3 9,2		5,3 9,2	5,3 9,2	5,3 9,2		5,3 9,2
5	6		7		8	9		10	11
	5,3 9,2 17,4		5,5 9,6 17,4		5,19 9,6 17,20	5,3 9,2 17,4		5,5 9,6 17,4	5,3 9,2 17,4
6	5	6			7		8		9
	5,3 9,10 17,4	5,5 9,6 17,4			5,15 9,2 17,48		5,15 9,2 17,48		5,15 9.2 17,48
7		5		6		7			8
		5,11 9,10 17,12 33,8		5,11 9,10 17,12 33,8		5,11 9,10 17,12 33,8			5,11 9,10 17,12 33,8
8	4		5		6			7	8
	5,3 9,6 17,12 33,8		5,13 9,10 17,12 33,8		5,11 9,10 17,12 33,8			5,11 9,10 17,12 33,8	5,5 9,18 17,12 33,8
9			5			6			7
			5,7 9,6 17,4 33,8 65,16			5,7 9,10 17,12 33,8 65,16			5,13 9,10 17,12 33,8 65,16
10		4		5			6		7
		69,5 73,10 81,4 97,8 129,16		5,3 9,10 17,12 33,8 65,16			5,7 9,18 17,12 33,8 65,16		5,7 9,18 17,12 33,8 65,16

TABLE 2
MORE EXAMPLES OF DCC LINEAR CONGRUENTIAL GRAPHS

degree	size	diam	const	size	diam	const
4	5^5	9	6,2 26,5			
5	$3*2^5$	4	13,7 25,10 49,4			
6	2^7	4	5,13 9,2 17,4	3^7	6	10,6 28,9 4,4
8	$3*2^5$	3	13,3 25,2 49,20 1,8			
9	2^7	3	5,13 9,10 17,4 33,24 65,16	$3*2^{10}$	5	13,5 25,18 49,28 97,24 193,16

We would like to point out that other symmetric graphs such as hypercubes, star graphs, and pancake graphs admit a similar construction.

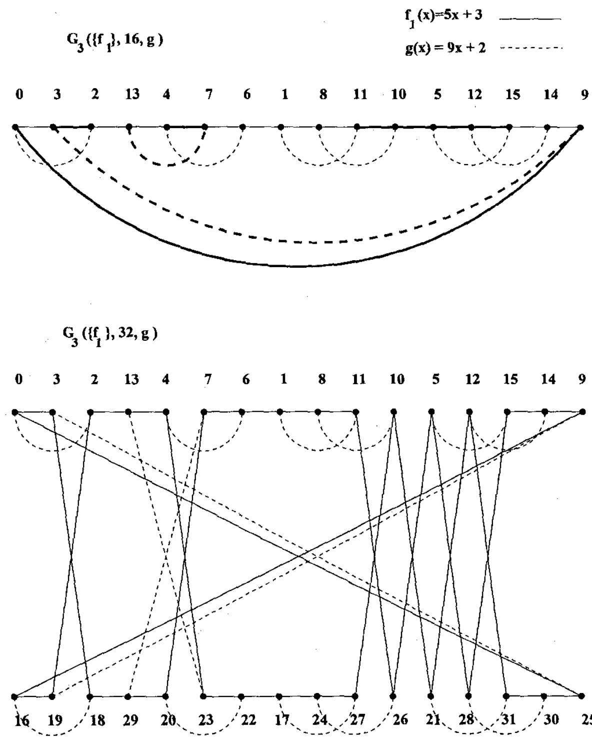

Fig. 3. Construction of $G_3(\{f_1\},\ 16,\ g)$ from $G_3(\{f_1\},\ 32,\ g)$.

4 CONCLUSIONS

The DCC linear congruential graphs presented in this paper form a very interesting family of graphs. Unlike de Bruijn graphs, they are defined for both odd and even degrees, are regular and of maximum connectivity for even degrees. They can be defined for any order which contains a multiple factor. Graph $G_t(F,\ n)$ is a proper subgraph of $G_{t+1}(F,\ n,\ f_{t+1})$ and $G_t(F,\ 2^p)$ can be constructed from two copies of $G_t(F,\ 2^{p-1})$. Thus, DCC linear congruential graphs satisfy the extensibility requirements in network design. As seen from the tables, they have many more vertices than graphs of the same diameter and degree produced by any other general construction. Furthermore, a DCC linear congruential graph is Hamiltonian and can be decomposed into a very small number of edge-disjoint cycles, which reminds a useful property of hypercubes. Thus, the DCC linear congruential graphs should be considered as an alternative for interconnection network designs.

The problem of obtaining a better upper bound on the diameter of DCC linear congruential graphs is a very interesting open problem that requires further studies. It seems to us that any good upper bound on the diameter will require a use of number theoretical properties of the constants of the linear functions.

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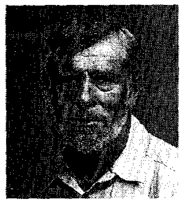
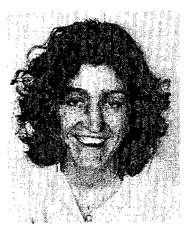


Jaroslav Opatrny received his first degree from Charles University, Prague, Czech Republic, in 1968, and the PhD in computer science from the University of Waterloo in 1975. From 1975 to 1977, he was an assistant professor at the University in Alberta, Edmonton, Canada. In 1977 he joined the Computer Science Department at Concordia University, Montreal, Canada, where he is presently an associated professor. His main interests are network models, algorithms, and graph theory.



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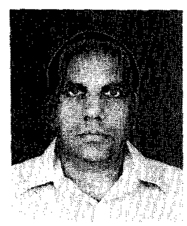

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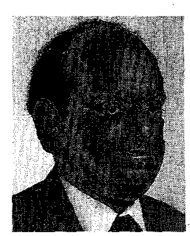

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Dr. Thulasiraman has published extensively on different aspects of electrical network theory, graph theory, and design and analysis of algorithms. He has co-authored two graduate level text books: *Graphs, Networks and Algorithms* and *Graphs: Theory and Algorithms*. He was an associate editor of the *IEEE Transactions on Circuits and Systems* and is currently a regional editor of the *Journal of Circuits, Systems and Computers*. He was technical program chair of the IEEE International Symposium on Circuits and Systems in 1993. His current research interests are in graph theory, parallel and distributed computations, operations research and computational graph theory with applications in VLSI design automation, computer networks, communication network planning, etc.