Incremental Distance and Diameter Sequences of a Graph: New Measures of Network Performance

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Abstract—**Motivated by their application in network topology design, two new measures of network performance, namely, the incremental distance sequence and the incremental diameter sequence, are introduced. These sequences can be defined for both vertex deletions and edge deletions. A complete characterization of the vertex-deleted incremental distance sequence is presented. Proof of this characterization is constructive in nature. A condition for the feasibility of an edge-deleted incremental distance sequence and a procedure for realizing such a sequence are given. Interrelationships between the elements of incremental distance sequences and incremental diameter sequences are studied. Using these results it is shown that a graph having a specified diameter and specified maximum increase in diameters for deletions of vertex sets of given cardinalities can be designed.**

Index Terms—**Connectivity, diameter, distance, fault-tolerant design, graph theory, interconnection network, network topology, reliability, shortest paths.**

I. Introduction

Recent advances in technology have made possible interconnection of a large number of computing elements to form an integrated multiprocessor system with processing, control, and information being distributed among these elements. Vulnerability, which is a measure of the ability of the system to withstand node or edge faults, and maximum routing delay are among the key considerations in the design of the topology of a multiprocessor system. Several measures of vulnerability have been defined and designs based on these measures have been presented in the literature. A commonly used measure of vulnerability is the maximum number of processors that can fail simultaneously without disabling any fault-free processor from communicating messages to every other fault-free processor. So connectivity of the graph underlying a multiprocessor system can be used as a measure of vulnerability of the system. The maximum delay suffered by a message is measured by the maximum length of the shortest paths between all pairs of elements and thus diameter can be used as a measure to evaluate the maximum delay in routing. Several important results relating to designs involving diameter and connectivity have been reported in the literature. For example see [1]–[4].

Whereas the maximum delay in routing is an important consideration in the design of the topology of a multiprocessor system, an equally important concern is the maximum increase that may occur in routing delay when processors or links fail. Chung and Garey [5] and Peyrat [6] have presented bounds on the maximum possible increase in diameter when some nodes or edges are deleted from a graph. Boesch *et al.* [7] have defined and studied a new measure of performance based on the minimum number of nodes or edges to be deleted to increase the diameter of a graph. Reddy *et al.* [8] have constructed a class of dense digraphs with minimum diameter and maximum connectivity. These graphs have the interesting property that the increase in their diameters is minimal when a certain number of edges and/or vertices is deleted. This work is based on the notion of t-deleted diameter introduced in [9]. Most recently, Sengupta *et al.* [10] have presented an interesting topology for which the maximum increase in diameter is again minimal unless the number of faulty processors is equal to the fault tolerance of the graph. This topology is based on the one proposed earlier by Pradhan [11]. In related works, Dolev *et al.* [12] and Broder *et al.* [13] have studied the issue of finding good fault-tolerant routings, that is, routings that keep the diameter of the surviving route graph small for any set of faults of a given cardinality.

Note that the maximum increase in the diameter, when some nodes or edges fail, is a global information. Suppose that the diameter is large, and that the network is neither dense nor sparse. In this case, it may so happen that the failure of some nodes or edges may not increase the diameter; but, that the routings between the vertices in the vicinity of the failed nodes or edges may be affected in such a way that the resulting reroutings introduce considerable delay when compared to the original routings. This is the case of a local disturbance having no effect on a global parameter. For example, in the graph of Fig. 1, when w is deleted the diameter does not change, but the distance between v and x changes from 2 to 3. In general, in many networks, the amount of communication between the nearer nodes will be higher than that between the farther nodes. Hence, measuring the effect of local disturbance becomes important, although it cannot be measured by a global parameter such as increase in diameter. In the case of distances, the maximum of the increases between every pair

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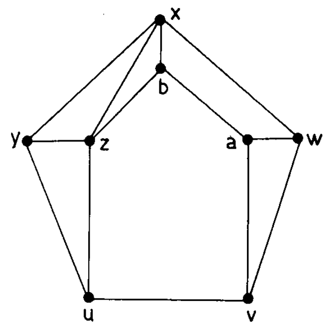

Fig. 1. Example illustrating the effect of local disturbance.

of nodes, when a fixed number of nodes or edges fails, gives a measure of the effect of the disturbance. The vulnerability of a network can be regarded to be smaller if these maximums for different sizes of node or edge failures are small. As such, these maximums can be regarded as vulnerability measures. Motivated by the above considerations, in this paper we formally define and initiate the study of two new measures of network performance, namely, the incremental distance and the diameter sequences of a graph. These sequences can be defined for both vertex deletions and edge deletions. Several results and constructions relating to these sequences are presented. The paper is organized as follows.

In Section II, we present certain preliminary results which form the basis of the results in subsequent sections. A complete characterization of the incremental distance sequence for vertex deletion is presented in Section III. The proof of this characterization is constructive in nature. The incremental distance sequence for edge deletion is studied in Section IV. Fairly general sufficient conditions for such a sequence to be realizable by a graph are also presented. Relationships between the elements of these two incremental distance sequences are given in Section V. Finally, in Section VI, relations between the incremental distance and the diameter sequences are studied. It is also shown that a graph having a specified diameter and specified maximum increases in diameter for deletion of vertex sets of given cardinalities can be designed. Section VII concludes the paper. To conserve space, we state certain results without proof. Proofs may be found in [14]. In the rest of this section, we present the basic terminology used in this paper.

We consider only simple graphs. For general notations we follow [15].

Let $G = (V, E)$.

Let $d(u, v) = d(u, v; G) =$ distance between u and v,

$d(G) =$ diameter of $G = \text{Max}\,\{d(u, v) | u, v \in V\}$

$N(W) = \{u \in V - W | (u, w) \in E$ for some $w \in W\}$, where $W \subseteq V$,

$N'(F) = \{u | (u, v) \in F$ for some $v \in V\}$, where $F \subseteq E$,

$a_i = a_i(G) = \text{Max}_{|V_i|=i}\{d(u, v; G - V_i) - d(u, v) | u, v \in V - V_i\}$,

$b_i = b_i(G) + \max_{|E_i|=i}\{d(u, v; G - E_i) - d(u, v)\}$,

$d_i = \text{Max}_{|V_i|=i}\{d(G - V_i)\}$,

$t_i = \text{Max}_{|E_i|=i}\{d(G - E_i)\}$,

$\delta =$ minimum degree in G,

$k =$ connectivity of $G = \text{Min}\,\{i | a_i = \infty\}$ and

$k' =$ edge connectivity of $G = \text{Min}\,\{i | b_i = \infty\}$.

$A = (a_1, a_2, \cdots, a_{k-1})$ is called the vertex-deleted incremental distance sequence,

$B = (b_1, b_2, \cdots, b_{k'-1})$ is called the edge-deleted incremental distance sequence,

$D = (d_1, d_2, \cdots, d_{k-1})$ is called the vertex-deleted diameter sequence, and

$T = (t_1, t_2, \cdots, t_{k'-1})$ is called the edge-deleted diameter sequence.

Note that a_i and b_i denote increments, whereas d_i and t_i represent the actual diameters. In the graph given in Fig. 1, we have $k = k' = 3$, $d = 3$, $A = \{1, 2\}$, $B = \{3, 3\}$, $D = \{3, 4\}$, and $T = \{4, 4\}$. To get $a_1 = 1$, note that $d(v, x; G - w) = 3$. To get a_2 and d_2 use $d(a, u; G - b - v) = 4$. The distance $d(u, v; G - (u, v)) = 4$ fixes b_1 and t_1.

Let $H = K_k$ and $V(H) = \{v_1, v_2, \cdots, v_k\}$, where K_k is the complete graph on k vertices, and $J = K_{k'}$.

By CHAIN$_i$ we mean the graph constructed as follows. Let each G_j, $1 \le j \le i$, be a distinct copy of K_k. Introduce all possible edges between the vertices of G_j and G_{j+1} for $j = 1, 2, \cdots, i - 1$. The resulting chain of K_k's is the graph CHAIN$_i$. G_1 and G_i are called the *endblocks* of this CHAIN and i is the *length* of the CHAIN. If i is odd, then $G_{(i+1)/2}$ is called the *middle block* of the CHAIN. If i is even then both $G_{i/2}$ and $G_{(i/2)+1}$ are middle blocks. ECHAIN$_i$ is defined similarly starting with $K_{k'}$.

In drawing the figures, we follow the convention given below to simplify the diagrams. A rectangle always represents a K_k when we consider vertex deletions, and a $K_{k'}$ when we consider edge deletions. A curly line between two rectangles, or between a vertex and a rectangle, represents all possible edges between the two. An arrow with a label i, from a vertex v, implies that v is adjacent to all the vertices in the CHAIN that contains the vertex u_i.

II. Preliminary Results

In this section, we show that in order to calculate the increase in distances in a subgraph, it is enough to consider the pairs of vertices belonging to the neighborhood of the deleted vertices or edges, instead of all the pairs of vertices.

Note 1: Since we consider only simple graphs, $a_i \ge 0$ and $b_i \ge 1$, for all i.

Note 2: If G is complete then A is a sequence of zeros and B is a sequence of ones. But the converse is not true, since for any $K_n - x$, $n \ge 4$, A and B have the same property. (Note: $K_n - x$ is the graph obtained from K_n after deleting any one edge.)

Note 3: If G is complete then D is a sequence of ones followed by a single zero; T is a sequence of twos.

Henceforth, (a) *we shall consider only graphs which are not complete,*

(b) *when we consider* a_i *or* d_i, $1 \le i \le k - 1$, *and*

(c) *when we consider* b_i *or* t_i, $1 \le i \le k' - 1$.

We now have the following result.

Theorem 1: The sequences A, B, D, and T are all monotonically nondecreasing.

Theorem 2: Suppose $|V_i| = i$ and $d(u, v; G - V_i) - d(u, v) + a_i$. Then there exist vertices b and c in $N(V_i)$ such that $d(b, c; G - V_i) = d(b, c) + a_i$.

Proof: If $a_i = 0$, there is nothing to prove. Let $a_i > 0$. Let P be a shortest path from u to v in G. P passes through at least one vertex of V_i, since otherwise $d(u, v)$ is not altered in $G - V_i$. Let b and c be the first and last vertices in P which are in $N(V_i)$. Clearly, $b \neq c$.

Suppose $d(b, c; G - V_i) < d(b, c) + a_i$.

Then

$$\begin{aligned} &d(u, v; G - V_i) \\ &\le d(u, b; G - V_i) + d(b, c; G - V_i) + d(c, v; G - V_i) \\ &< d(u, b) + d(b, c) + a_i + d(c, v) \\ &< d(u, v) + a_i. \end{aligned}$$

This contradiction completes the proof.

Theorem 3: Suppose $|E_i| = i$ and $d(u, v; G - E_i) = d(u, v) + b_i$. Then there exist vertices b and c in $N'(E_i)$ such that $d(b, c; G - E_i) = d(b, c) + b_i$.

Corollary 1: The above two theorems indicate that the definitions of a_i and b_i can be simplified as follows.

$$a_i = \mathrm{Max}_{|V_i|=i}\{d(u, v; G - V_i) - d(u, v) | u, v \in N(V_i)\}$$

and

$$b_i = \mathrm{Max}_{|E_i|=i}\{d(u, v; G - E_i) - d(u, v) | u, v \in N'(E_i)\}.$$

This definition reduces to a large extent the amount of computation needed to determine a_i and b_i.

III. Characterization of Vertex-Deleted Incremental Distance Sequences

A monotonically nondecreasing sequence $x_1, x_2, \cdots, x_{k-1}$, with $x_1 \geq 0$, is said to be a *feasible A-sequence*, if there exists a simple graph G whose A-sequence is the given sequence. Similar definitions hold for B, D, and T-sequences. The feasible A-sequence is characterized in this section.

Theorem 4: Any monotonically nondecreasing sequence $x_1, x_2, \cdots, x_{k-1}$ with $x_1 \geq 0$ is a feasible A-sequence.

Proof: The proof is by construction. If $x_i = 0$ for all i, then $G = K_{k+1}$ is the required graph.

Suppose some $x_i > 0$.

Let $R = \{r_i\}$ be the distinct nonzero elements in A, in the ascending order. Let $|R| = r$.

Let $F = \{f_i\}$ where $f_i = \mathrm{Min}\{j | x_j = r_i\} =$ the first position in A where the ith distinct nonzero element occurs.

For an easier understanding of the proof, we consider two cases.

Case 1: Let $r = 1$.

Consider the graphs $H (= K_k)$ and CHAIN_x where $x = 2r_1 + 1$, as defined in Section I. Join $v_1, v_2, \cdots, v_{f_1}$ to every vertex in the CHAIN. Join $v_{f_1+1}, v_{f_1+2}, \cdots, v_k$ to the vertices of the endblocks of CHAIN. Finally, add a vertex u and connect it to every vertex in H. The resulting graph is the required graph G. Note that the diameter of G is 2, since v_1 is adjacent to every other vertex. Let u_1 be any one vertex in G_{r_1+1}, the middle block in the CHAIN.

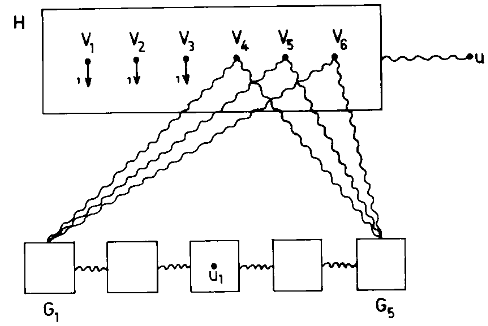

Fig. 2. Example for construction described in Theorem 4 (case 1).

We illustrate the construction by an example. For the sequence $\{0, 0, 2, 2, 2\}$, we have $k = 6$, $R = \{2\}$, $F = \{3\}$, and $x = 5$. The graph G is given in Fig. 2.

We now show that the graph G realizes the sequence $\{x_1, x_2, \cdots, x_{k-1}\}$ as its A-sequence.

First, note that deleting u, or any subset of vertices of the CHAIN, does not affect the distances between the remaining vertices. Consider next the following sets of vertices $\{v_1, v_2, \cdots, v_{f_1}\}$, $\{v_{f_1+1}, v_{f_1+2}, \cdots, v_k\}$, $V(G_i)$, $1 \leq i \leq x$. If two vertices x and y belong to the same set, they are not only similar in G, but also have the same adjacencies except that x is adjacent to y, and y is adjacent to x. (Note: Two vertices are similar in G, if there is an automorphism of G taking one to the other [15].) Hence, any path passing through x and not through y can be transformed into a path passing through y and not through x, just by replacing x by y. Hence, deleting any proper subset of any of the sets considered is not going to affect the distance between any two of the remaining vertices. Since each G_i has k vertices, none of the G_i's will be deleted completely from G, and this implies that it is enough to consider the effects of the deletion of vertices in H.

Since $\{v_i\} \cup N(v_i)$ contains $N(v_j)$, for $i < j$, deletion of v_j will have no effect on the distances if some v_i, $i < j$, is not deleted. Hence, only sets of the form $\{v_1, v_2, \cdots, v_i\}$ are to be considered for deletion. Now, the discussion in the previous paragraph indicates that only the set $V' = \{v_1, v_2, \cdots, v_{f_1}\}$ is to be considered.

Now, $d(u_1, u; G - V') = r_1 + 2 = r_1 + d(u_1, u)$. From the construction it is obvious that this increase is maximum and hence

$$a_i = \begin{cases} 0 & \text{for } i < f_1 \\ r_1 & \text{for } f_1 \leq i < k. \end{cases}$$

This completes the proof for Case 1.

Case 2: Let $r \geq 2$.

The construction here is a generalization of the one given above. For each i, $1 \leq i \leq r$, a distinct CHAIN is to be constructed, whose length is determined as follows; the increase in length in the previous stages has to be taken into account while determining the length of the CHAIN for the current stage. Determine the functions *needed, length*, and *available* recursively as follows.

{$needed_i$ =the amount of distance to be increased at the ith stage, taking into account the usable distance increase from the previous stages.
$length_i$ =length of the CHAIN for the ith stage.
$available_i$ =the increase in distance available at the end of the ith stage, which can be used in the next stage.}
$available_0$ =0;
For $i = 1$ to r do
$needed_i$ $= r_i - available_{i-1}$;
$length_i$ $= 2\ (needed_i)+1$;
$available_i$ $= \mathrm{Max}\{available_{i-1}, needed_i\}$;
end-of-for.

By finite induction it can be shown that for $1 \le i \le k-1$, $0 < available_i \le r_i$ and $0 < needed_i \le r_i$.

Consider the graphs H and $\mathrm{CHAIN}_i = \mathrm{CHAIN}_{length_i}$, $1 \le i \le r$. For each i, $1 \le i \le r$, join $v_1, v_2, \cdots, v_{f_i}$ to every vertex of CHAINi, and join $v_{f_i+1}, v_{f_i+2}, \cdots, v_k$ to all the vertices of the endblocks of CHAINi. Let u_i by any one vertex in the middle block of CHAINi, and let w_i be any one vertex in one endblock of CHAINi. The resulting graph of diameter 2 is G.

To illustrate the ideas, let $\{0, 0, 1, 4, 4, 5\}$ be the given sequence. Here $k = 7$, $R = \{1, 4, 5\}$, $r = 3$, and $F = \{3, 4, 6\}$. The graph is shown in Fig. 3.

Proceeding as in Case 1, we get

a) $a_i = 0$ for $1 \le i \le f_1 - 1$,

b) $a_i = a_j$ if $f_x \le i < j < f_{x+1}$ for some x, where f_{r+1} is taken as k, and

c) only the sets $\{v_1, v_2, \cdots, v_{f_1}\}$, $\{v_1, v_2, \cdots, v_{f_2}\}$, $\cdots$, $\{v_1, v_2, \cdots, v_{f_r}\}$ are to be considered to get $a_{f_1}, a_{f_2}, \cdots, a_{f_r}$. We will be done once we show that $a_{f_i} = x_{f_i}$, for all i.

It is easy to verify that $a_{f_1} = r_1$, since $d(u_1, u_2; G - V_1) = 2 + r_1$ where $V_i = \{v_1, v_2, \cdots, v_{f_i}\}$, and this represents a maximum increase in the distances.

Suppose we have proved $a_j = x_j$ for $j < f_i$, for some $i \ge 2$. We shall now prove $a_{f_i} = r_i$.

From the definition we have

$$
\begin{aligned}
available_1 &= r_1 = d(u_1, w_1; G - V_1)\\
available_{i-1} &= \mathrm{Max}\,\{d(u_x, w_x; G - V_{i-1})\}\\
&= \mathrm{Max}\,\{d(u_x, w_x; G - (V_i - v_{f_i})\}.
\end{aligned}
$$

Suppose the above maximum occurs for $x = p$. Cleary $p < i$. Now $d(u_p, u_i; G-V_i) = d(u_p, w_p; G-V_i) + d(w_p, w_i; G-V_i) + d(w_i, u_i; G - V_i) = available_{i-1}+2+needed_i = 2+r_i$.

It is clear that for any $q < i$, $d(u_q, u_i; G - V_i) \le 2 + r_i$, by the maximality attached to p.

If $q > i$, then $d(u_q, u_i; G - V_i) = 2 + needed_i \le 2 + r_i$.

If $x, y < i$, then $d(u_x, u_y; G - V_i) \le 2 + r_{i-1} < 2 + r_i$. The method of construction indicates that the maximum increase in distance occurs between the u_j's, and hence $a_{f_i} = r_i$. This completes the proof for Case 2 and the theorem.

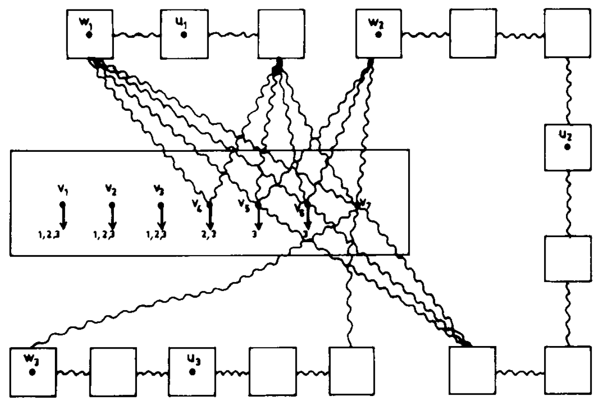

Fig. 3. Example for construction described in Theorem 4 (case 2).

IV. Edge-Deleted Incremental Distance Sequences: Conditions for Feasibility

If $b_1 = 1$, let p be the greatest integer such that $b_p = 1$. If $b_1 > 1$, let $p = 0$.

If $(x, y) \in E$, then there exist at least p edge disjoint paths of length 2 between x and y, since $b_p = 1$.

If $d(x, y) = 2$, then there exist at least $p + 1$ edge disjoint paths of length ≤ 3 between x and y, since by Theorem B [16] the maximum number of such paths is equal to the minimum number of edges to be deleted to make the distance between x and y as > 3, and this minimum is $\ge p + 1$.

Let $X = \{x_1, x_2, \cdots, x_{k'-1}\}$ be any monotonically nondecreasing sequence. Let $R = \{r_i\}$ be the increasing sequence of distinct elements of X, which are > 1. Let $F = \{f_i\}$, where $f_i = \min\{j | x_j = r_i\}$. Note that the definitions of R and F are the same as before, except that here in R we consider only elements > 1, instead of > 0. Let $|R| = r$. Now $p = f_i - 1$.

Theorem 5: Any monotonically nondecreasing sequence $X = \{x_1, x_2, \cdots, x_{k'-1}\}$, with $x_1 \ge 1$ and $x_2 \ge 2$, is a feasible B-sequence.

Proof: Here $p = 0$ or 1.

Consider a $J = K_{k'}$, to start with. For each i, $1 \le i \le r$, do the following. Form an ECHAIN of length $r_i + 1$. Identify one middle block of this ECHAIN with J. Introduce a new vertex u_i, and join it to all the vertices of an endblock in the ECHAIN and to any f_i number of vertices of the other endblock.

Finally, introduce a new vertex u and join it to every vertex in J. The resulting graph G has X as its B-sequence. The proof is straightforward and hence omitted.

For $X = \{1, 3, 3, 4\}$, we have $R = \{3, 4\}$, $F = \{2, 4\}$, and the graph G is as given in Fig. 4. In this example, the deletion of the two edges drawn individually at u_1 increases the distance between u_1 and v_1 from 1 to 4. Similarly, the deletion of the four edges drawn individually at u_2 increases the distance between u_2 and v_2 from 1 to 5. These deletions account for r_1 and r_2.

Note that the above construction will work for any sequence with all $x_i \le 3$, but will not work if $x_2 = 1$ and some $x_i \ge 4$, since in this case, the increase in distance is 2 in $G - (u_i, w) -$

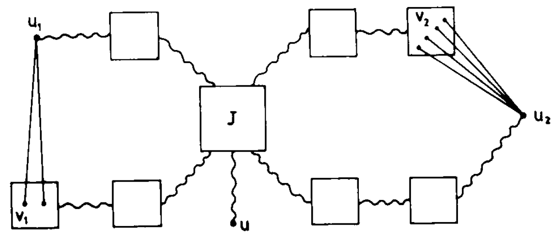

Fig. 4. Example illustrating Theorem 5.

(u_i, z) where w and z belong to different endblocks of the ECHAIN of length $x_i + 1$.

Now, we derive some relations between the elements of the B-sequence, based on the value of p.

Lemma 1: Let $b_2 = 1$. Let $E' = \{e_1, e_2, \cdots, e_q\}$ be any set of edges in any graph G which forms a star. Let $q < k'$. Then $d(x, y; G - E') \le d(x, y) + 3$ for all x, y.

Proof: Let $e_i = (u, v_i)$ for $1 \le i \le q$. Here $p \ge 2$, since $b_2 = 1$. In view of Theorem 3, it is enough to consider x and y in $\{v_i\} \cup \{u\}$.

Let $x = v_i$ and $y = v_j, i \ne j$. If $(x, y) \in E$ then $d(x, y; G - E') = 1$. If $(x, y) \notin E$, then $d(x, y) = 2$. Since E' is a star with the center u different from x and y, at most two edge disjoint paths of length ≤ 3 between x and y are destroyed in $G - E'$. Since there are at least $p + 1 \ge 3$ such paths of length ≤ 3 between x and y, we get $d(x, y; G - E') \le 3 = d(x, y) + 1$.

Let $x = u$ and $y = v_i$. Let $u, w_1, w_2, \cdots, w_s = v_i$ be a shortest path between x and y in $G - E'$. If $(w_1, v_i) \in E$, then $d(x, y; G - E') = 2$. If $(w_1, v_i) \notin E$, then $d(w_1, v_i) = 2$ and as before not all the (≥ 3) paths of length ≤ 3 between w_1 and v_i are destroyed in $G - E'$. Hence, $d(w_1, v_i; G - E') \le 3$. This gives $d(x, y; G - E') \le 4 = d(x, y) + 3$.

Lemma 2: Let $\{b_i\}$ be the B-sequence of G. If $b_2 = 1$, then $b_{p+1} = r_1 \le 3$.

Proof: Let $E' = \{e_i = (u_i, v_i) | 1 \le i \le p + 1\}$ be such that $d(u, v; G - E') = d(u, v) + b_{p+1}$ for some $u, v \in N'(E')$.

Suppose every pair of edges of E' has a common end vertex. Then E' forms a star or a K_3. If E' forms a K_3, then between any two vertices of this K_3, there exists a path of length 2 which does not pass through the third vertex, since $p \ge 2$. This gives $b_{p+1} = 1$ by Theorem 3, a contradiction. Hence, E' forms a star and the result follows from Lemma 1.

Suppose there exist two edges, say e_1 and e_2, in E', which do not share a common vertex. Let $E'' = E' - e_1$. Since $|E''| = p$, $d(x, y; G - E'') \le d(x, y) + 1$ for all x, y. Also, since e_2 does not share a vertex with e_1, not all the $(\ge p)$ paths of length 2 between u_1 and v_1 (ends of e_1) are destroyed in $G - E''$. Hence, for all x, y, $d(x, y; G - E'' - e_1) \le d(x, y; G - E'') + 1 \le d(x, y) + 2$. This implies $b_{p+1} = 2$ and the proof is complete.

Lemma 3: Let $\{b_i\}$ be the B-sequence of G and let $b_3 = 1$. Then $b_{p+2} \le b_{p+1} + 1$.

Proof: Here $p \ge 3$. Let $E' = \{e_i = (u_i, v_i) | 1 \le i \le p + 2\}$ be such that $d(u, v; G - E') = d(u, v) + b_{p+2}$ for some $u, v \in N'(E')$.

Suppose every pair of edges in E' has a common end vertex. In this case, E' forms a star and by Lemma 1, $b_{p+2} \le 3 \le b_{p+1} + 1$, since $b_{p+1} \ge 2$.

Suppose there exist two edges, say e_1 and e_2, in E', which do not share a common vertex.

Case 1: Not all the $(\ge p)$ paths of length 2 between u_1 and v_1 are destroyed in $G - E''$, where $E'' = E' - e_1$.

Here $d(x, y; G - E'' - e_1) \le d(x, y; G - E'') + 1$ and this gives $b_{p+2} \le b_{p+1} + 1$.

Case 2: Not all the $(\ge p)$ paths of length 2 between u_2 and v_2 are destroyed in $G - \{E' - e_2\}$. This case is similar to the previous one.

Case 3: All the $(\ge p)$ paths of length 2 between u_1 and v_1, and between u_2 and v_2, are destroyed in $G - E'$.

In order to destroy all the above mentioned paths, each one of the edges in $\{e_3, e_4, \cdots, e_{p+2}\}$ should be present in exactly one of the edge disjoint paths of length 2 between u_1 and v_1, and in exactly one such path between u_2 and v_2. Clearly all these edges must be adjacent to both e_1 and e_2. These requirements can be met only by two edges. In other words, $p \le 2$, a contradiction. Hence, this case is impossible and the proof is complete.

Lemma 4: If $b_4 = 1$ and $1 \le j \le p$, then $b_{p+j} \le b_{p+j-1} + 1$.

Proof: Let $E' = \{e_i | 1 \le i \le p + j\}$ be such that $d(u, v; G - E') = d(u, v) + b_{p+j}$ for some $u, v \in N'(E')$. The proof is easy in the first two cases of Lemma 3.

Case 3: Edges e_1 and e_2 of E' do not share a common vertex, and all the $(\ge p)$ paths of length 2 between u_1 and v_1, and between u_2 and v_2, are destroyed in $G - E'$.

This is possible only if at least p edges from $\{e_3, e_4, \cdots, e_{p+j}\}$ are adjacent to e_1 and such that no two of them have a common vertex other than u_1 or v_1. A similar statement holds true with respect to e_2 also. Since at most two edges can satisfy these requirements and also be adjacent to both e_1 and e_2, we have $p + j \ge 2(p - 2) + 4$. That is $p \le j$.

Since $j \le p$ by our assumption, we have $p = j$ and two edges, say $e_3 = (u_1, u_2)$ and $e_4 = (v_1, v_2)$ (without loss of generality) are adjacent to both e_1 and e_2. Also, $p - 2$ of the edges of $E' - \{e_1, e_2, e_3, e_4\}$ are adjacent to e_1 and not to e_2, and each one of these $p - 2$ edges is on a distinct path of length 2 between u_1 and v_1. The remaining $p - 2$ edges of E' are adjacent to e_2 and not to e_1, and each one is on a distinct path of length 2 between u_2 and v_2. Since every edge in $E' - \{e_1, e_2\}$ meets e_1 or e_2, there exists no $e_i \in E' - \{e_1, e_2\}$ such that $\{e_1, e_2, e_i\}$ are independent. Also, since e_3 and e_4 are on paths of length 2 between u_1 and v_1, we have $(u_1, v_2), (u_2, v_1) \in E - E'$. Note that these conclusions have been arrived at starting with a pair of independent edges e_1 and e_2 in E'. The same conclusions hold good for every pair of independent edges in E', and this implies that no three edges in E' form an independent set of edges.

Suppose e_5 meets e_1 at u_1. Then we cannot have an edge $e_i \in \{e_6, e_7, \cdots, e_{2p}\}$ such that e_i is incident on u_2 and is not adjacent to e_5, since otherwise $\{e_4, e_5, e_i\}$ will form a set of three independent edges. On the other hand, if some e_i, $i \ge 6$, meets e_2 at u_2, and is adjacent to e_5 also, then considering the independent edges e_3 and e_4, we see that both e_5 and e_i are on a single path of length 2 between u_1 and u_2, and hence not all the paths of length 2 between u_1 and u_2 are destroyed in $G - E'$, a contradiction. Hence, any e_i, $i \geq 6$, which meets e_2 can meet e_2 only at v_2. This in turn implies that any e_i, $i \geq 6$, meeting e_1, meets it only at u_1.

Suppose $e_s = (u_1, w_s)$ and $e_t = (v_2, w_t)$ are in $E' - \{e_1, e_2, e_3, e_4\}$. If $w_s \neq w_t$, then considering the independent edges e_i and e_t, we get that (u_1, w_t) and (v_2, w_s) are both in E', and $(w_s, w_t) \in E - E'$, since $(u_1, v_2) \in E - E'$. Hence, the edges of E' form a union of two stars, each with p edges, having distinct central vertices, but the same set of end vertices. Also, there is an edge in $E - E'$ joining any two of these end vertices. The presence of these edges and the edge (u_1, v_2) in $E - E'$ imply that the vertices u and v mentioned in the beginning of the proof are such that one is the center of a star described above and the other is an end vertex of that star. Let $u = u_1$ and $v = v_1$ without loss of generality.

Let $u, x_1, x_2, \cdots, v$ be a shortest path in $G - E'$. Since $b_{p+j} > 1$, we have $d(u, v; G - E') \geq 3$ and this implies $(x_1, v) \notin E$ and $d(x_1, v) = 2$. Since E' is the union of two stars as described before, at most four edge disjoint paths of length ≤ 3 can be destroyed by deleting E' (see Fig. 5). As there are at least $p + 1 \geq 5$ such paths between x_1 and v, $d(x_1, v; G - E') \leq 3$. This gives $d(u, v; G - E') \leq 4$ and hence $b_{p+j} \leq 3$. Since $b_{p+j-1} \geq 2$, we get $b_{p+j} \leq b_{p+j-1} + 1$ and the proof is complete.

It can be shown that (for example, see Fig. 6) when $p = 2$ or 3, b_{2p} is not bounded above by $p + 2$, as in the case for $p \geq 4$, and hence Lemma 4 cannot be extended to the cases where $p = 2$ or 3. Combining the three lemmas we have the following.

Theorem 6: If $p = 2$ then $b_3 \leq 3$. If $p = 3$ then $b_4 \leq 3$ and $b_5 \leq 4$. If $p \geq 4$ then $b_{p+j} \leq j + 2$ for $1 \leq j \leq p$.

The above result shows that the number of ones in B has some influence on the next few elements of B, and, if the number of ones is more, greater is the influence. Note that we have seen that when $p = 1$, it has no effect on any b_i, $i \geq 2$. The natural question is whether this influence spreads throughout B or does it stop somewhere? In case it stops at some stage, can the area of influence be determined? We show that, under certain circumstances, this influence does not spread after a certain stage.

Theorem 7: Let $X = \{x_1, x_2, \cdots, x_{k'-1}\}$ be a monotonically nondecreasing sequence such that $x_i = 1$ for $1 \leq i \leq p$, where $p \geq 2$, $x_{p+1} \geq 2$, and $x_{p^2-1} \leq 3$. Then X is a feasible B-sequence

Proof: Consider $J = K_{k'}$.

For each i, such that $r_i = 2$ or 3, proceed as in the construction given in the proof of Theorem 5.

For each i, such that $r_i > 3$, form an ECHAIN of length $r_i + 2$. Identify one middle block of this ECHAIN with J. Choose the smallest s such that

$$(s-1)s < f_i \leq s^2 \text{ or } s^2 < f_i \leq s(s+1).$$

In the first case, choose s vertices each, from the endblocks of the ECHAIN. In the second case, choose s vertices from one endblock and $s + 1$ vertices from the other. Introduce f_i edges between these $s + s$ or $s + (s + 1)$ vertices, in any way, as in a bipartite graph.

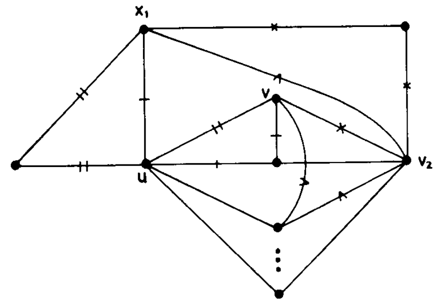

Fig. 5. Example used in proof of Lemma 4.

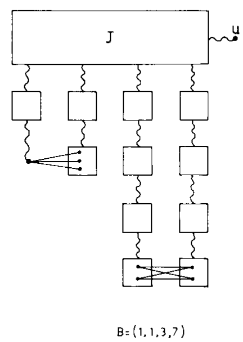

Fig. 6. Example illustrating the nonapplicability of the bound in Lemma 4 for the case $p = 2$ or $p = 3$.

As the final step, introduce a new vertex u and join it to every vertex in J. The resulting graph G is the required graph. An example is given in Fig. 6.

The B-sequence of G is determined as follows. From the construction it is obvious that only the Σf_i edges introduced joining the ends of the ECHAINS are to be considered for deletion.

Where $r_i = 2$ or 3, the length of the ECHAIN being 3 or 4 ensures $b_p = 1$ and $b_{f_i} \geq r_i$.

Suppose some $r_i > 3$. If $(x, y) \in E$, and x and y are in different endblocks of the corresponding ECHAIN, we shall show that there are at least p edge disjoint paths of length 2 between x and y. This will ensure $b_p = 1$. We assumed $x_{p^2-1} \leq 3$ precisely to facilitate this, as shown below.

Since $r_i > 3$ we have $f_i \geq p^2$ and hence $s \geq p$. Suppose $s + s$ vertices have been chosen at the endblocks of the corresponding ECHAIN (the case $s + (s + 1)$ can be treated similarly). Here $(s - 1)s < f_i \leq s^2$. Let H be the bipartite graph introduced on these $2s$ vertices using f_i edges. Clearly, the number of edge disjoint paths of length 2 between x and y, say q, is $\deg_H x + \deg_H y - 2$ since $(x, y) \in E$. When $f_i = p^2$, we have $s = p$, H is a complete bipartite graph, and hence $q = 2p - 2 \geq p$ since $p \geq 2$. Hence, let $f_i > p^2$. In this case, $s > p$, since we are having exactly $s + s$ vertices in H.

We have $f_i = \deg_H x + \deg_H y - 1 + h$ where h = number of edges in H having no end in $\{x, y\} \leq (s - 1)^2$. Hence, $f_i = q + 1 + h$. This gives $s \leq q + 1$, since $s(s - 1) < f_i$ and $h \leq (s - 1)^2$. Now $p < s$ implies $p \leq q$, the required result.

Since deletion of any $f_i - 1$ edges from H can increase the distance between any two vertices by at most two, it follows that X is the B-sequence of G.

V. Interrelations Between Incremental Distance Sequences

It is natural to expect that $a_i \geq b_i$ for $1 \leq i \leq k'$, since the deletion of one vertex can delete more than an edge. But this need not be so. In this section, a lower bound for a_i is derived in terms of b_i. This brings out the relation between the A and B-sequences of a graph.

Theorem 8: For $i \geq 1$, $a_i \geq b_i - 2$ and this bound can be achieved.

Proof: Since $a_i \geq 0$, there is nothing to prove if $b_i \leq 2$. Hence, let $b_i \geq 3$. Let $d(u, v; G - E_i) = d(u, v) + b_i$ where $|E_i| = i$, $E_i \subset E$, and $u, v \in N'(E_i)$.

Case 1: Let $(u, v) \notin E_i$.

Obviously $(u, v) \notin E$. Let $(u, u_1), (v, v_1) \in E_i$. Let V' be a set of vertices formed by taking one end vertex of every edge in E_i, such that $u_1, v_1 \in V'$ and $u, v \notin V'$. We have $|V'| \leq i$ and

$$\begin{aligned} d(u, v) + b_i &= d(u, v; G - E_i) \\ &\leq d(u, v; G - V') \leq d(u, v) + a_i. \end{aligned}$$

Hence, $a_i \geq b_i$ in this case.

Case 2: Let $(u, v) \in E_i$.

Since $b_i \geq 3$, we have $d(u, v; G - E_i) = 1 + b_i > 3$. Let $u, u_1, u_2, \cdots, u_x, v$ be a shortest path from u to v in $G - E_i$.

Suppose $(u, u_x) \notin E$. Then $d(u, u_x) = 2$ and $d(u, u_x; G - E_i) = b_i$. Choose V' as having one end vertex of every edge in E_i such that $v \in V'$ and $u, u_x \notin V'$. Since $|V'| \leq i$, we have $d(u, u_x; G - E_i) \leq d(u, u_x; G - V') \leq 2 + a_i$. Hence, $a_i \geq b_i - 2$ in this subcase.

Suppose $(u, u_x) \in E$. Since $d(u, v; G - E_i) > 3$, this implies $(u, u_x) \in E_i$. If, for every vertex t with $(v, t) \in E - E_i$, the edge (u, t) is in E, then $(u, t) \in E_i$ and this will imply $|E_i| \geq \delta > k'$, a contradiction. Hence, there exists a vertex t such that $(v, t) \in E - E_i$ and $(u, t) \notin E$, $d(u, t; G - E_i) \geq b_i$, as otherwise $d(u, v; G - E_i) < 1 + b_i$. Now choose V' as having one end vertex from each edge of E_i such that $u, t \notin V'$ and $u_x, v \in V'$. Since $|V'| \leq i$, $b_i \leq d(u, t; G - E_i) \leq d(u, t; G - V') \leq 2 + a_i$ and hence $a_i \geq b_i - 2$.

This proves $a_i \geq b_i - 2$ in all the cases and for all i. Examples achieving this bound can be constructed.

Theorem 9: If $a_1 = 0$ then $b_{p+1} = 2$.

Proof: Let $E' = \{e_i | 1 \leq i \leq p + 1\}$ be such that $d(u, v; G - E') = d(u, v) + b_{p+1}$ for some $u, v \in N'(E')$.

If E' forms a K_3 or has two independent edges then the proof is the same as in Lemma 2. Hence, let E' form a star. Let $e_i = (s, w_i)$, $1 \leq i \leq p+1$. Since $p + 1 < \delta$, there exists a vertex w_{p+2} such that $(s, w_{p+2}) \in E - E'$. Now $a_1 = 0$ implies that $d(w_i, w_j; G - s) \leq 2$ for $1 \leq i \leq p + 2$ and hence $d(w_i, w_j; G - E') \leq 2$. Since $u, v \in \{w_i\} \cup \{s\}$, we get $b_{p+1} = 2$.

VI. Relations Between Incremental Distance and Diameter Sequences

The relationships between all the four sequences are considered in this section. These relations allow us to translate the result obtained for one sequence to that for the other. We use the results $d_i \leq (n - i - 2)/(k - i) + 1$ and $t_i \leq (i + 1)d + i$ proved by Chung and Garey [5].

Theorem 10: Any monotonically nondecreasing sequence $x_1, x_2, \cdots, x_{k-1}$ with $x_1 \geq x \geq 2$ is a D-sequence of a graph G of diameter x.

Proof: Let G' be the graph constructed in Theorem 4, for the sequence $x - 2, x_1 - 2, x_2 - 2, \cdots, x_{k-1} - 2$. Then the diameter of $G' - \{v_1, \cdots, v_{j+1}\} = x_j$, and the connectivity of G' is $k + 1$. Let $G = G' - v_1$. G is the required graph.

Theorem 11: Let G be a given graph. Then

$$d_i - d \leq a_i \leq d_i - 2, \text{ and } t_i - d \leq b_i \leq t_i - 1.$$

Proof: For the first part, consider the ends of a diameter. The second part follows since $d(u, v; G - V_i) \leq d_i$.

Corollary 2: Suppose $t_p = 2$. If $p = 2$, then $t_3 \leq 5$. If $p = 3$ then $t_4 \leq 5$ and $t_5 \leq 6$. If $p \geq 4$ then $t_{p+j} \leq j + 4$ for $j \leq p$.

Proof: Since $t_2 = 2$, we have $d = 2$. This gives $t_p - 2 \leq b_p \leq t_p - 1$ and hence $b_p = 1$. Now, Theorem 7 and the inequality $t_i - d \leq b_i$ imply the required result.

Note 3: The sequence 2, 2, 2, 2, 4, 7 is not a feasible T-sequence.

The constructions given for $\{b_i\}$ cannot be modified for $\{t_i\}$, unlike in the case of $\{a_i\}$ and $\{d_i\}$, because of the following. In the case of vertex deletion, there exist subsets of vertices V_i, such that $|V_i| = i$, $V_1 \subset V_2 \subset V_3 \cdots$, deletion of V_i changes some distance by a_i, and, most importantly, the diameter of the constructed graph is 2. None of these properties hold good in the construction given for $\{b_i\}$.

Corollary 3: $t_i \leq d_i + d$ and $t_i \leq \lfloor (n - i - 2)/(k - i) \rfloor + d + 1$.

Proof: $t_i - d - 2 \leq b_i - 2 \leq a_i \leq d_i - 2 \leq \lfloor (n - i - 2)/(k - i) \rfloor + 1 - 2$.

Corollary 4: $a_i \leq \lfloor (n - i - 2)/(k - i) \rfloor - 1$ and $b_i \leq \min\{(i + 1)d + i + 1, \lfloor (n - i + 2)/(k - i) \rfloor + d\}$

Proof: These results follow from $a_i \leq d_i - 2$ and $b_i \leq t_i - 1$.

VII. Conclusion

Motivated by their application in network topology design, we have introduced two new measures of network performance, namely, the incremental distance sequence and the diameter sequence. These sequences are defined for both vertex deletions and edge deletions. We have established several results relating to these sequences. We have given general procedures to design networks having prescribed incremental distance and diameter sequences. The number of vertices in these networks is linearly related to the number of distinct nonzero elements in the sequences. In practical designs, such nondistinct elements will be very few in number. For such cases, the size of the networks designed by our procedures will be considerably smaller than the sizes suggested by the examples given in this paper.

Several interesting open problems remain. It is not clear if polynomial time algorithms are possible to determine the elements of the incremental sequences. It will be interesting to evaluate the several interconnection topologies presented in the literature using these new measures of fault tolerance, and present new designs based on these measures along with certain other parameters like connectivity and the number of vertices.

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References

[1] J. C. Bermond, J. Bond, M. Paoli, and C. Peyrat, "Graphs and interconnection networks: Diameter and vulnerability," in *Surveys in Combinatorics*, E. K. Lloyd, Ed., LMS Lecture Note Series 82, 1983, pp. 1-30.

[2] M. Imase, T. Soneoka, and K. Okada, "Connectivity of regular directed graphs with small diameters," *IEEE Trans. Comput.*, vol. 34, pp. 267-273, 1985.

[3] U. Schumacher, "An algorithm for construction of a k-connected graph with minimum number of edges and quasiminimal diameter," *Networks*, vol. 14, pp. 63-74, 1984.

[4] V. Krishnamoorthy, K. Thulasiraman, and M. N. S. Swamy, "Minimum order graphs wth specified diameter, connectivity and regularity," *Networks*, to be published.

[5] F. R. K. Chung and M. R. Garey, "Diameter bounds for altered graphs," *J. Graph Theory*, vol. 8, pp. 511-534, 1984.

[6] C. Peyrat, "Diameter vulnerability of graphs," *Discrete Appl. Math.*, vol. 9, pp. 245-250, 1984.

[7] F. T. Boesch, F. Harary, and J. A. Kabell, "Graphs as models of communication network vulnerability: Connectivity and persistence," *Networks*, vol. 11, pp. 57-63, 1981.

[8] S. M. Reddy, J. G. Kuhl, S. H. Hosseini, and H. Lee, "On digraphs with minimum diameter and maximum connectivity," in *Proc. 20th Annu. Allerton Conf. Commun., Contr., Comput.*, 1982, pp. 1018-1026.

[9] J. G. Kuhl, "Fault diagnosis in computing networks," Ph.D. dissertation, Dep. Comput. Sci., Univ. of Iowa, 1980.

[10] A. Sengupta, A. Sen, and S. Bandyopadhyay, "On an optimally fault-tolerant network architecture," *IEEE Trans. Comput.*, vol. C-36, pp. 619-623, 1987.

[11] D. K. Pradhan, "Fault-tolerant multiprocessor link and bus network architecture," *IEEE Trans. Comput.*, vol. C-34, pp. 33-45, 1985.

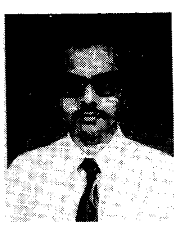
[12] D. Dolev, J. Y. Halpern, B. Simons, and H. R. Strong, "A new look at fault-tolerant network routing," *Inform. Computat.*, vol. 72, pp. 180-196, 1987.

[13] A. Broder, D. Dolev, M. Fischer, and B. Simons, "Efficient fault-tolerant routings in networks," in *Proc. 16th Annu. ACM Symp. Theory Comput.*, 1984, pp. 536-541.

[14] V. Krishnamoorthy, K. Thulasiraman, and M. N. S. Swamy, "Incremental distance and diameter sequences of a graph: New measures of fault-tolerance," Tech. Rep., Dep. Elec. Comput. Eng., Concordia Univ., Montreal, P.Q., Canada, 1988.

[15] F. Harary, *Graph Theory*. Reading, MA: Addison-Wesley, 1972.

[16] G. Exoo, "On a measure of communication network vulnerability," *Networks*, vol. 12, pp. 405-409, 1982.



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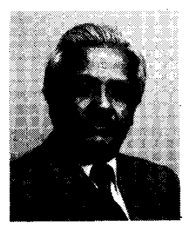

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