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Identifying codes of cycles with odd orders[☆,☆☆]

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Abstract

The problem of the r-identifying code of a cycle C_n has been solved totally when n is even. Recently, S. Gravier et al. give the r-identifying code for the cycle C_n with the minimum cardinality for odd n, when $n \geq 3r + 2$ and $\gcd(2r + 1, n) \neq 1$. In this paper, we deal with the r-identifying code of the cycle C_n for odd n, when $n \geq 3r + 2$ and $\gcd(2r + 1, n) = 1$.

1. Introduction

Let $G = (V(G), E(G))$ be a simple, connected, undirected graph and $r \geq 1$ be an integer. Given a vertex $x \in V$, we define $B_r(x) = \{y : d(x, y) \leq r\}$ where $d(x, y)$ denotes the distance of the shortest path between x and y in G. For a subset S of V, we say that S r-covers x if $B_r(x) \cap S \neq \emptyset$. We say that a subset S r-separates two distinct vertices u and v if and only if $B_r(u) \cap S \neq B_r(v) \cap S$. An r-identifying code of G is a set $S \subseteq V$ which r-covers all the vertices of G and r-separates any pair of distinct vertices of G.

If for any pair of distinct vertices $u, v \in V$, $u \neq v$, we have $B_r(u) \neq B_r(v)$, then V itself is an r-identifying code. Therefore, the associated optimization problem is to find the minimum cardinality of such a code, which we denote by $M_r(G)$.

The concept of identifying code was first introduced in [8]. An illustration comes from fault diagnosis in multiprocessor systems. We want to find the faulty vertices correctly if at

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most one vertex is wrong. For this purpose, we select some vertices and use them to test their r-neighborhoods (i.e., the vertices at distance at most r). If there is something wrong within this neighborhood, the testing vertex sends a signal about this malfunction. Our aim is to distinguish the faulty vertex from others only by using the information that is obtained from the vertices which we have selected.

Now, the optimization problem of determining an identifying code with minimum cardinality in a graph has been proved to be NP-hard [3]. Many people have focused on the study of identifying codes in some restricted classes of graphs, for example [1,2,4,5]. In this paper, we are interested in finding the minimum cardinality of an identifying code in cycles which has already been investigated in [1–7,9,10].

2. Previous results and lemmas

A cycle C_n for $n \geq 3$ is a graph $(V(C_n), E(C_n))$ with $V(C_n) = \{v_i : i \in \mathbb{Z}_n\}$ and $E(C_n) = \{v_i v_{i+1} : i \in \mathbb{Z}_n\}$ where $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. For n even, Bertrand et al. give the following theorem in [2].

Theorem 1 (*Bertrand et al. [2]*). *For all $r \geq 1$, we have $M_r(C_{2r+2}) = 2r + 1$ and $M_r(C_n) = \frac{n}{2}$ for $n \geq 2r + 4$ even.*

In [7], Gravier et al. define a graph $C'_{(n,r)}$ on the vertex set $\{v_i : i \in \mathbb{Z}_n\}$ such that, for all $i \in \mathbb{Z}_n$, $v_{i-r}v_{i+r+1}$ is an edge of $C'_{(n,r)}$. By using such a graph, they proved Theorem 2.

Theorem 2 (*Gravier et al. [7]*). *For all $r \geq 1$ and $n \geq 2r + 3$ odd, we have*

$$\frac{n+1}{2} + \frac{\gcd(2r+1, n) - 1}{2} \leq M_r(C_n) \leq \frac{n+1}{2} + r.$$

For large n, Gravier et al. give the following result.

Lemma 3 (*Gravier et al. [7]*). *Let $r \geq 1$, n be an odd integer such that $n \geq 3r + 2$, and S be an edge cover set of $C'_{(n,r)}$ such that all the vertices of C_n are r-covered by S. Then S is an r-identifying code of C_n.*

By using the above lemma, they get the theorems below.

Theorem 4 (*Gravier et al. [7]*). *Let $r \geq 1$, n be an odd integer such that $3r + 2 \leq n \leq 4r + 1$, and S be an edge cover set of $C'_{(n,r)}$. Then S is an r-identifying code of C_n.*

Theorem 5 (*Gravier et al. [7]*). *Let $r \geq 1$, n be an odd integer such that $\gcd(2r + 1, n) = 1$, and $4r + 5 \leq n \leq 8r + 1$. Then any edge cover set of $C'_{(n,r)}$ is an r-identifying code of C_n.*

Theorem 6 (*Gravier et al. [7]*). *Let $r \geq 1$, n be an odd integer such that $n \geq 3r + 2$ and $\gcd(2r+1, n) \neq 1$. Then there exists an optimal edge cover set of $C'_{(n,r)}$ which is an r-identifying code of C_n.*

Proposition 7 (*Daniel [4] and Gravier et al. [7]*). *$M_1(C_5) = 3$; $M_1(C_n) = \frac{n+3}{2}$ for all $n \geq 7$, n odd; $M_r(C_{2r+3}) = \lfloor \frac{4r+6}{3} \rfloor$ for all $r \geq 1$; $M_r(C_{4r+3}) = 2r + 3$.*

Table 1 shows all of the results concerning the r-identifying code of cycle C_n with odd n.

In this paper, we will deal with the r-identifying code with odd n such that $n > 8r + 1$ and $\gcd(2r + 1, n) = 1$.

Table 1
The value of $M_r(C_n)$

odd n	$n > 8r+1$	$4r+5 \le n \le 8r+1$	$n = 4r+3$	$3r+2 \le n \le 4r+1$	$2r+5 \le n < 3r+2, r \ge 4$	$n = 2r+3$
$\gcd(2r+1, n) \ne 1$	$\frac{n+1}{2} + \frac{\gcd(2r+1,n)-1}{2}$		$2r+3$	$\frac{n+1}{2} + \frac{\gcd(2r+1,n)-1}{2}$	?	$\lfloor \frac{2n}{3} \rfloor$
$\gcd(2r+1, n) = 1$	?	$\frac{n+1}{2}$				

3. Main results

In this section, we will state and give the proof of our main results.

Theorem 8. *Let $r \ge 1$, n be an odd integer such that $n \ge 3r+2$, and $\gcd(2r+1, n) = 1$. If $n = 2m(2r+1)+1$ or $n = (2m+1)(2r+1)+2r$ for $m \ge 1$, then $M_r(C_n) = \frac{n+1}{2} + 1$; otherwise $M_r(C_n) = \frac{n+1}{2}$.*

Proof. We will prove the result according to the following cases:

Case 1: If $n = 2m(2r+1)+x$ with $m \ge 1$ and $3 \le x \le 2r-1$, then $M_r(C_n) = \frac{n+1}{2}$.

We set $V(C'_{(n,r)}) = \{w_i = v_{i(2r+1)} : i \in \mathbb{Z}_n\}$ and $E(C'_{(n,r)}) = \{(w_i, w_{i+1}) : i \in \mathbb{Z}_n\}$. Let $S = w_0 \cup \{w_i : i \text{ is odd}\}$. Obviously, S is an edge cover set of $C'_{(n,r)}$ with order $\frac{n+1}{2}$. Choose $\{v_i, v_{i+1}, \ldots, v_{i+2r}\}$ to be any arbitrary $2r+1$ consecutive vertices. Since $0 \le i \le n-1$, we set $i = j(2r+1)+k$ where $0 \le j \le 2m-1$ and $0 \le k \le 2r$ or $j = 2m$, $0 \le k \le x-1$. Let $l = \lfloor \frac{x}{2r+1-x} \rfloor$. Next, we will show that S r-covers all the vertices of C_n with the following two subcases:

Subcase 1.1 $l \ge 1$. In this subcase, if $l = \frac{x}{2r+1-x}$, then $2r+1-x | x$, $2r+1-x | 2r+1$ and $2r+1-x | n$, which contradict $\gcd(2r+1, n) = 1$ for $3 \le x \le 2r-1$. So, we have $l < \frac{x}{2r+1-x}$. And we have $l(2r+1-x) < x < (l+1)(2r+1-x)$.

If j is even with $0 \le j \le 2m-2$, then $v_{i+(2r+1-k)} = v_{(j+1)(2r+1)} = w_{j+1} \in S$ for $k > 0$ and $v_{i+(2r+1-x)} = v_{(j+(2m+1))(2r+1)} = w_{j+(2m+1)} \in S$ for $k = 0$. If $j = 2m$, then $v_{i+(x-k)} = v_0 = w_0 \in S$.

If j is odd with $1 \le j \le 2m-1$, then $v_{i+2(2r+1-x)-k} = v_{(j+2(2m+1))(2r+1)} = w_{j+2(2m+1)} \in S$ for $k \le 2(2r+1-x)$ and $v_{i+(2r+1)-k+(2r+1-x)} = v_{(j+(2m+2))(2r+1)} = w_{j+(2m+2)} \in S$ for $k > (2r+1-x)$.

Subcase 1.2 $l = 0$. Let $l' = \lfloor \frac{2r+1-x}{x} \rfloor$; then $l' \ge 1$ and $l'x < 2r+1-x < (l'+1)x$ since $\gcd(2r+1, n) = 1$.

If j is even with $0 \le j \le 2m-2$, then $v_{i+(2r+1-k)} = v_{(j+1)(2r+1)} = w_{j+1} \in S$ for $k > 0$ and $v_{i+(2r+1-x)} = v_{(j+(2m+1))(2r+1)} = w_{j+(2m+1)} \in S$ for $k = 0$. If $j = 2m$, then $v_{i+(x-k)} = v_0 = w_0 \in S$.

If j is odd with $1 \le j \le 2m-1$, then $v_{i+2(2r+1)-(l'+2)x-k} = v_{(j+2m(l'+2)+2)(2r+1)} = w_{j+2m(l'+2)+2} \in S$ for $k \le 2(2r+1)-(l'+2)x$ and $v_{i+4(2r+1)-k-(2l'+3)x} = v_{(j+2m(2l'+3)+4)(2r+1)} = w_{j+2m(2l'+3)+4} \in S$ for $k > 2(2r+1)-(l'+2)x$.

According to the above discussion, we have $M_r(C_n) = \frac{n+1}{2}$ by Lemma 3.

Case 2: If $n = 2m(2r+1)+1$ for $m \ge 1$, then $M_r(C_n) = \frac{n+3}{2}$.

Case 3: If $n = (2m + 1)(2r + 1) + x$ for $m \geq 1$ and $2 \leq x \leq 2r - 2$ or $m = 0$ and $r + 1 \leq x \leq 2r - 1$, then $M_r(C_n) = \frac{n+1}{2}$.

Case 4: If $n = (2m + 1)(2r + 1) + 2r$, then $M_r(C_n) = \frac{n+3}{2}$.

The proofs of Case 2, Case 3 and Case 4 are similar and are not detailed here.

Owing to the above discussion, we get the theorem. ■

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References

[1] N. Bertrand, I. Charon, O. Hudry, A. Lobstein, 1-Identifying codes on trees, Australasian Journal of Combinatorics 31 (2005) 21–35.
[2] N. Bertrand, I. Charon, O. Hudry, A. Lobstein, Identifying and locating-dominating codes on chains and cycles, European Journal of Combinatorics 25 (2004) 969–987.
[3] I. Charon, O. Hudry, A. Lobstein, Minimizing the size of an identifying or locating-dominating code in a graph is NP-hard, Theoretical Computer Science 290 (2003) 2109–2120.
[4] M. Daniel, Codes identifiants, in: Mémoire pour le DEA ROCO, Université Joseph Fourier, Grenoble, France, June 2003.
[5] M. Daniel, S. Gravier, J. Moncel, Identifying codes in some subgraphs of the square lattice, Theoretical Computer Science 319 (2004) 411–421.
[6] A. Frieze, R. Martin, J. Moncel, M. Ruszinkó, C. Smyth, Codes identifying sets of vertices in random networks, Discrete Mathematics 307 (2007) 1094–1107.
[7] S. Gravier, J. Moncel, A. Semri, Identifying codes of cycles, European Journal of Combinatorics 27 (2006) 767–776.
[8] M.G. Karpovsky, K. Chakrabarty, L.B. Levitin, On a new class of codes for identifying vertices in graphs, IEEE Transaction on Information Theory 44 (1998) 559–611.
[9] D.L. Roberts, F.S. Roberts, Locating sensors in paths and cycles: The case of 2-identifying codes, European Journal of Combinatorics 29 (2008) 72–82.
[10] http://www.infres.enst.fr/lobstein/bibLOCDOMetID.html.