Today

- Binary addition
- Representing negative numbers
Binary Addition

Consider the following binary numbers:

0 0 1 0 0 1 1 0 0 0 1 0 1 0 1 1

How do we add these numbers?
Binary Addition

00100110
00101011

1
Binary Addition

0 0 1 0 0 1 1 0
0 0 1 0 1 0 1 1

And we have a carry now!
Binary Addition

\[ \begin{array}{cccccc}
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
\end{array} \]

\[ \begin{array}{c}
0 & 0 & 1 \\
\end{array} \]

And we have a carry again!
Binary Addition

00100110
00101011

0001

and again!
Binary Addition

0 0 1 0 0 1 1 0
0 0 1 0 1 0 1 1

1 0 0 0 1
Binary Addition

00100110
00101011

010001

One more carry!
Binary Addition

\[ \begin{array}{ccccccc}
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
\end{array} \]
Binary Addition

Behaves just like addition in decimal, but:

- We carry to the next digit any time the sum of the digits is 2 (decimal) or greater.
Negative Numbers

So far we have only talked about representing non-negative integers

• What can we add to our binary representation that will allow this?
Representing Negative Numbers

One possibility:

- Add an extra bit that indicates the sign of the number
- We call this the "sign-magnitude" representation
Sign Magnitude Representation

+12 0 0 0 0 1 1 0 0
Sign Magnitude Representation

+12 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0

-12 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0
Sign Magnitude Representation

+12 \quad 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0

-12 \quad 1 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0

What is the problem with this approach?
Sign Magnitude Representation

What is the problem with this approach?

- Some of the arithmetic operators that we have already developed do not do the right thing
Sign Magnitude Representation

Operator problems:
• For example, we have already designed a counter (that implements an ‘increment’ operation)

-12 1 0 0 0 1 1 0 0
Sign Magnitude Representation

Operator problems:

-12

10001100

Increment
Sign Magnitude Representation

Operator problems:

-12

1 0 0 0 1 1 0 0

Increment

1 0 0 0 1 1 0 1
Sign Magnitude Representation

Operator problems:

-12  1 0 0 0 1 1 0 0

Increment

-13  1 0 0 0 1 1 0 1

!!!!

Andrew H. Fagg: Embedded Real-Time Systems: Binary Arithmetic
Representing Negative Numbers

An alternative:
(a little intuition first)

0

0 0 0 0 0 0 0 0

Decrement
Representing Negative Numbers

An alternative:
(a little intuition first)

0

0 0 0 0 0 0 0 0

Decrement

1 1 1 1 1 1 1 1
Representing Negative Numbers

An alternative:
(a little intuition first)

Define this as

00000000

Decrement

-1111111111
Representing Negative Numbers

A few more numbers:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0 0 0 0 0 0 0 1 1</td>
</tr>
<tr>
<td>2</td>
<td>0 0 0 0 0 0 1 0</td>
</tr>
<tr>
<td>1</td>
<td>0 0 0 0 0 0 0 1</td>
</tr>
<tr>
<td>0</td>
<td>0 0 0 0 0 0 0 0</td>
</tr>
<tr>
<td>-1</td>
<td>1 1 1 1 1 1 1 1</td>
</tr>
<tr>
<td>-2</td>
<td>1 1 1 1 1 1 1 0</td>
</tr>
<tr>
<td>-3</td>
<td>1 1 1 1 1 1 0 1</td>
</tr>
</tbody>
</table>
Two’s Complement Representation

In general, how do we take the additive inverse of a binary number?
Two’s Complement Representation

In general, how do we take the additive inverse of a binary number?

- Invert each bit and then add ‘1’
Two’s Complement Representation

Invert each bit and then add ‘1’

5 \[\rightarrow\] \[0 0 0 0 0 1 0 1\] Two’s complement

-5 \[\rightarrow\] \[1 1 1 1 1 0 1 1\]
Two’s Complement Representation

Now: let’s try adding a positive and a negative number:

\[
\begin{array}{c}
12 & \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \\
+ & \quad + \\
-5 & \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \\
\end{array}
\]
Two’s Complement Representation

Now: let’s try adding a positive and a negative number:

\[ \begin{align*}
12 & \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \\
+ & \quad + \\
-5 & \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \\
\end{align*} \]

\[ \downarrow \]

\[ \begin{align*}
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \\
\end{align*} \]
Two’s Complement Representation

Now: let’s try adding a positive and a negative number:

```
12      0 0 0 0 1 1 0 0
+      +
-5    1 1 1 1 1 0 1 1
```

```
7      0 0 0 0 0 1 1 1
```
Two’s Complement Representation

Two’s complement is used for integer representation in today’s processors
Two’s Complement Representation

Two’s complement is used for integer representation in today’s processors.

One oddity: we can represent one more negative number than we can positive numbers.
Implementing Subtraction

How do we implement a ‘subtraction’ operator?
(e.g., $A - B$)
Implementing Subtraction

How do we implement a ‘subtraction’ operator?
(e.g., A – B)

- Take the 2s complement of B
- Then add this number to A
Representing Fractions

Floating point representations are expensive:

- Require many bits
- Either require specialized hardware or long functions to compute mathematical operations
A Low-Cost Alternative: Fixed Point Representations

“w.f” fixed point:
• w bits to represent the whole number (including the sign)
• f bits to represent the fraction
A Low-Cost Alternative: Fixed Point Representations

“w.f” fixed point:
• We are representing values in units of $2^{-f}$

So: 5.3 fixed point
• 5 bits for whole
A Low-Cost Alternative: Fixed Point Representations

5.3 fixed point (fits in an int8_t)
- 5 bits for whole
- 3 bits for fraction

What can we represent with this?
A Low-Cost Alternative: Fixed Point Representations

What can we represent with 5.3 fixed point?

- 5 bits for whole: 15 ... -16
- 3 bits for fraction: units of 1/8th
## Fixed-Point Example

<table>
<thead>
<tr>
<th>Fixed Point</th>
<th>Value</th>
<th># of eighths</th>
</tr>
</thead>
<tbody>
<tr>
<td>00000 000</td>
<td>0.0</td>
<td>0 eighths</td>
</tr>
<tr>
<td>00000 001</td>
<td></td>
<td></td>
</tr>
<tr>
<td>00000 100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>00001 000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>00101 010</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Fixed-Point Example

<table>
<thead>
<tr>
<th>Fixed Point</th>
<th>Value</th>
<th># of eighths</th>
</tr>
</thead>
<tbody>
<tr>
<td>000000 000</td>
<td>0.0</td>
<td>0 eighths</td>
</tr>
<tr>
<td>000000 001</td>
<td>0.125</td>
<td>1 eighth</td>
</tr>
<tr>
<td>000000 100</td>
<td>0.5</td>
<td>4 eighths</td>
</tr>
<tr>
<td>000001 000</td>
<td>1.0</td>
<td>8 eighths</td>
</tr>
<tr>
<td>001010 010</td>
<td>5.25</td>
<td>42 eighths</td>
</tr>
</tbody>
</table>
Adding Fixed-Point Numbers

```c
int8_t a = 5; // 5/8
int8_t b = 10; // 10/8
int8_t c = a + b ???
```

$$5 \left( \frac{1}{8}s \right) + 10 \left( \frac{1}{8}s \right) = 15 \text{ what?}$$
Adding Fixed-Point Numbers

```c
int8_t a = 5;    // 5/8
int8_t b = 10;   // 10/8
int8_t c = a + b; // 15/8
```

$$5 \left( \frac{1}{8} s \right) + 10 \left( \frac{1}{8} s \right) = 15 \left( \frac{1}{8} s \right)$$

So: addition does the right thing
Multiplying Fixed-Point Numbers

```c
int8_t a = 5;    // 5/8
int8_t b = 10;   // 10/8
int8_t c = a * b ???
```

\[ 5 \left( \frac{1}{8} s \right) \times 10 \left( \frac{1}{8} s \right) = 50 \] what?
Multiplying Fixed-Point Numbers

```c
int8_t a = 5;    // 5/8
int8_t b = 10;   // 10/8
int8_t c = a * b ???
```

\[
5 \left( \frac{1}{8} s \right) \times 10 \left( \frac{1}{8} s \right) = 50 \left( \frac{1}{64} s \right)
\]

But: we need to keep things in 5.3 format
Multiplying Fixed-Point Numbers

```c
int8_t a = 5;    // 5/8
int8_t b = 10;   // 10/8
int8_t c = (a * b) >> 3; // 6/8
```

\[
5 \left( \frac{1}{8} s \right) \times 10 \left( \frac{1}{8} s \right) = 50 \left( \frac{1}{64} s \right) \approx 6 \left( \frac{1}{8} s \right)
\]
Dividing Fixed-Point Numbers

int8_t a = 20;    // 20/8
int8_t b = 7;     // 7/8
int8_t c = a / b  // ???

\[ 20 \left( \frac{1}{8} s \right) \div 7 \left( \frac{1}{8} s \right) = 2 \text{ What?} \]
Dividing Fixed-Point Numbers

int8_t a = 20;  // 20/8
int8_t b = 7;   // 7/8
int8_t c = a / b ???

\[ 20 \left( \frac{1}{8} s \right) \div 7 \left( \frac{1}{8} s \right) = 2 \ (1s) \]

But: we want to stay within the 5.3 format. And – note that we have lost information in the rounding!
Dividing Fixed-Point Numbers

```c
int8_t a = 20;    // 20/8
int8_t b = 7;     // 7/8
int8_t c = (a << 3) / b; // 160/7
```

\[
20 \left( \frac{1}{8} s \right) \div 7 \left( \frac{1}{8} s \right) = 22 \left( \frac{1}{8} s \right)
\]
Notes About the Book

The example code that the book gives tries to address some additional questions (but fails to be clear):

- In conversions from floating point to fixed-point, it catches errors when a floating point value is too small or too large to fit in the fixed point representation
- `assert(0)` just means that an error should be generated
Notes About the Book

• In the book, a “short” is 16 bits and a “long” is 32 bits.
• For many of the fixed-point examples, the fixed-point values fit in 16 bits.
• After we perform a mathematical operation, it is possible that the result will not fit within the 16 bits.
• So: all numbers are converted to 32 bits before the operation & the results are checked before converting back to 16 bits.