Reliability Evaluation of Multistage Networks Considering Replacement Time

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Abstract: In some recent papers, the reliability analyses for multistage interconnection networks (MIN) have been carried out with the assumption that the failed components are not repaired (replaced). However, in most practical cases a faulty component would be repaired (replaced) within a certain period of time. In this paper, we determine the *network reliability* of MINs with component failures and normal replacement time, and determine the maximum allowable replacement time (MART) for a given reliability. The use of approximate cut set method has been demonstrated for reliability calculation of the multistage network.

1 Introduction

Multiprocessor systems are becoming increasingly popular in solving large-scale compute bound problems. Multistage interconnection networks (MIN) have been found to be very cost effective for connecting a large number of processors and memories. MINs have also been used as ATM switching fabrics in the emerging Broadband Integrated Services Digital Networks (B-ISDN).

A MIN (see Figure 1) consists of several stages of small crossbar switching elements (SE). The successive stages are connected by a permutation function. A unique path MIN [1, 2] consists of a single path from any source to any destination. Failure of any SE or any link connecting the stages would remove the *full access* property of the network. The above problem can be solved by providing fault tolerance using one or a combination of the following methods.

- Purely software approach whereby error-detecting/correcting codes are used.
- Hardware/software fault tolerance approach where redundancy is obtained by using a combination of encoding and voting.
- Providing more than one (multiple) path to get from the source to the destination.

Multiple paths can be provided by having multiple links in parallel, having extra stages, or having multiple copies of the same network.

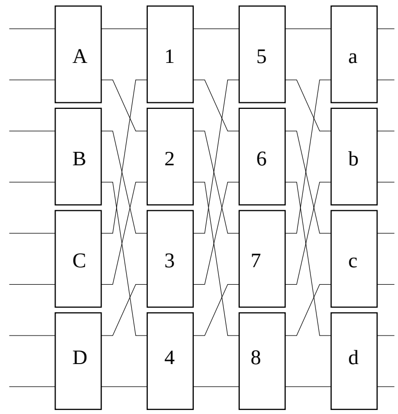

Figure 1: An 8×8 SEN+ network.

Several studies have been carried out to determine the reliability of multiple path MINs. Blake and Trivedi [3] have derived closed-form expressions for the time-dependent reliability of the 8×8 and 16×16 extra stage shuffle exchange network (SEN+). They have also derived a reliability lower bound that is useful for the analysis of large networks. The bounds have been substantially improved in [4].

Most of the reliability studies have been carried out with the assumption that the failed components are not repaired (replaced). However, in most practical cases a faulty component would be repaired (replaced) within a certain period of time. It is therefore important to evaluate the reliability of MINs by taking the repair (replacement) time into consideration. The *objectives* of this paper are as follows:

- To determine the *network reliability* of MINs with component failures and normal replacement time. Network reliability (also called the all-terminal reliability) addresses the probability that at least one path exists between each source and every destination.
- To show the *variation of reliability* with repair (replacement) time and determine the maximum allowable replacement time (MART) for a given reliability.

A description of the MIN is given in Section 2 followed by our solution approach in Section 3. The cut set method and the approximate solution technique along with the results are given in Sections 4 and 5 followed by conluding remarks in Section 6.

2 Network Description

The SEN+ [3] will be used as an example to demonstrate our approach of finding the reliability of multiple paths MINs by considering repair (replacement) time. To increase the reliability of the shuffle-exchange network (SEN), an SEN+ network is constructed by adding an extra stage to the SEN. The SEN+ connects N processors to N memories using $n = \log_2 N + 1$ stages of $N/2$ SEs per stage. A perfect shuffle permutation [5] is used to connect adjacent stages as shown in Figure 1. Each SE is a 2×2 crossbar allowing any input link to be connected to any one of the output links. A conflict arises if two packets at the two buffers of an SE request the same output link simultaneously. A SEN+ has two paths of constant length between any input-output pair. The SEs in the first and last

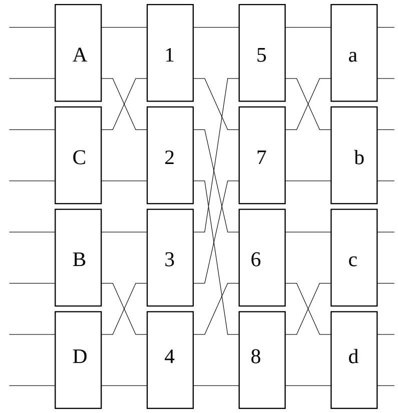

Figure 2: An 8×8 SEN+ network with the SEs in the first and third stage rearranged.

stage must *always be operational* to provide at least one path (full access) between every possible input-output pair.

3 Approach

The SEN or SEN+ system can not be analysed by simple series, parallel or a combination of these structures due to the complex operational logic. Additional techniques and modelling are required to determine the reliability of such systems. Most of these methods are formalized methods for transforming the complex topology or the operational logic of a Markov process into series-parallel combinations. The *conditional probability* approach is one of the methods to determine the reliability of a complex system. This approach sequentially reduces a complex system into subsystem structures that consists of only series-parallel connections and these subsystems are then combined together using the conditional probability method to get the reliability of the original complex system.

The conditional probability approach uses the relation

$$P(\textit{system success}) = P(\textit{system success given component } X \textit{ is good}) \times P(\textit{component } X \textit{ is good}) + P(\textit{system success given component } X \textit{ is bad}) \times P(\textit{component } X \textit{ is bad})$$

where X could be any component in a given system.

The SEs in Figure 1 can be rearranged as shown in Figure 2 to facilitate the visual understanding of the network. Applying the conditional probability approach, it can be shown that the reliability equation is given by

$$R_{\text{SEN+}}(t) = 2r^{12} + 4r^{14} - 8r^{15} + 3r^{16} \tag{1}$$

where r is the reliability of a single SE and is equal to $r = e^{-\lambda t}$ for constant failure rate.

3.1 Reliability evaluation in repairable systems

The time dependent probabilities for a system can be calculated using the relation:

$$[P'(t)] = [C][P(t)] \tag{2}$$

where $[C]$ is the coefficient matrix [6]. This coefficient matrix in the above equation is not a stochastic transitional probability matrix because the rows of this coefficient matrix instead of summing to unity actually sums to zero. The above equation can be solved by different methods. The probabilities calculated using the full coefficient matrix can not be used to calculate the reliability of a system as the success in case of reliability means staying out of the failed state. It should also be noted that it can be quite difficult to obtain general reliability expression for a more complex systems with repair as in the SEN+ network. The concept of absorbing state [6] can be used to calculate the reliability of a system using numerical solutions to solve the above set of differential equations. In the present situation, there are a total of 256 states for the eight components and by a careful examination of the states it could be shown that the system works only for 35 states. By deleting the columns and rows corresponding to the failed states from $[C]$, a reduced set of differential equations is obtained for the 35 working states as

$$[P_r'(t)] = [C_r][P_r(t)] \tag{3}$$

where P_r is a 35×1 vector containing the working states and $[C_r]$ is a 35×35 coefficient matrix obtained from $[C]$ using the absorbing state concept as mentioned above. These equations are solved with numerical method for a total of 8760 hours. The results for the cases with repair and without repair for different values of replacement time are presented in Figures 3 and 4. These results indicate that the replacement time has significant effect on the reliability when the failure rates are higher.

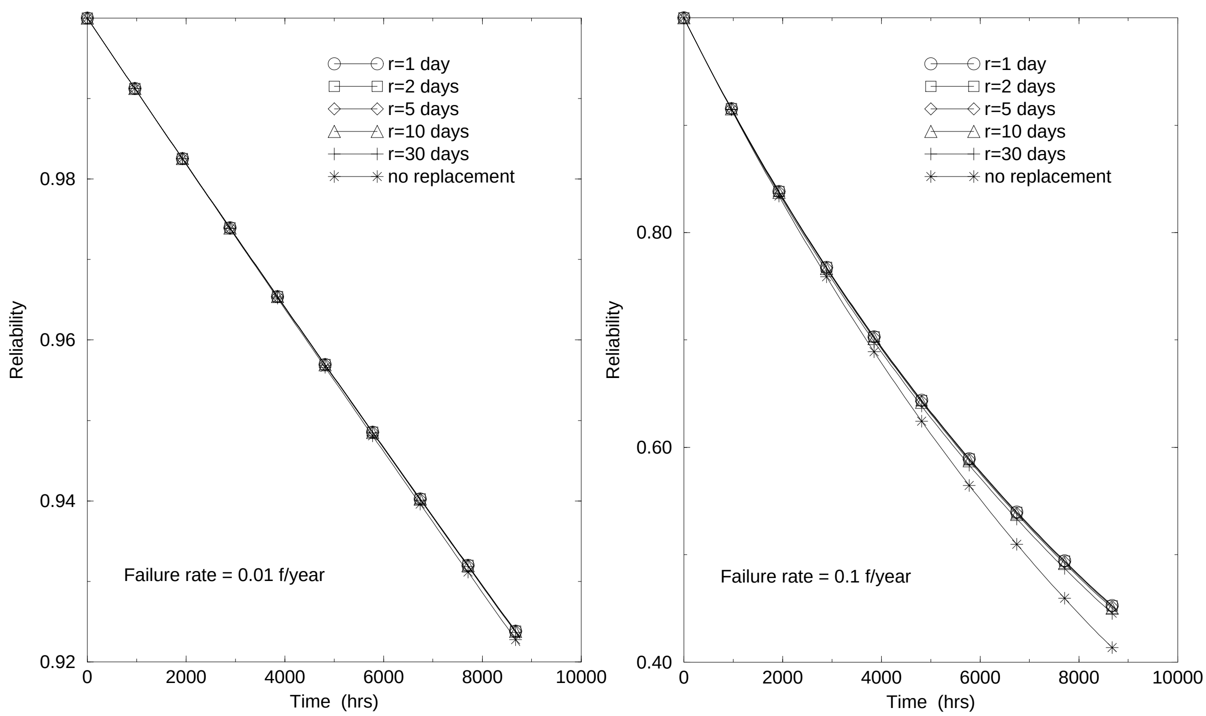

Figure 3: Reliability versus time for different values of replacement time and λ=0.01 and 0.1 failures/year.

4 Cut Set Method

The *cut set* method is another powerful tool for reliability evaluation of complex systems. The main advantages of this method are

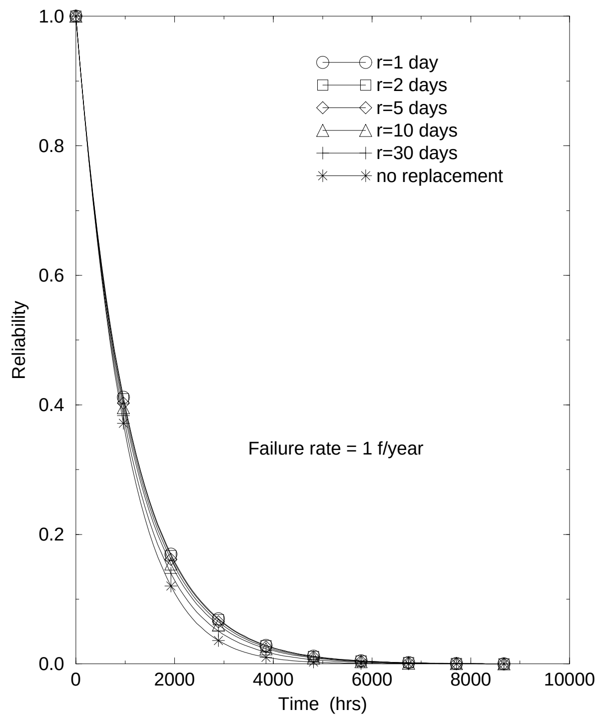

Figure 4: Reliability versus time for different values of replacement time and $\lambda = 1.0$ failures/year.

- the cut set algorithm can be implemented easily in a computer for fast and efficient solution for any general network
- cut sets are directly related to the modes of system failure, and hence identify the distinct and discrete ways in which a system may fail.

The minimal cut sets identified for a complex network must be combined together in order to evaluate system reliability. *All the components* of each minimal cut set *must fail* to make the system failure according to the definition of minimal cut sets. It means that the components of a given minimal cut set are connected *in parallel* as the system failure require the failure of all components in the minimal cut set. The failure probabilities of the components in a given minimal cut set can be combined using the principle of parallel system. The principle of series system can not be used for combining different minimal cut sets. The concept of union must be used to set the unreliability of the system. The minimal cutsets for the SEN+ have been deduced and are given by (A), (B), (C), (D), (a), (b), (c), (d), (1,2), (3,4), (5,7), (6,8), (1,7), (1,8), (2,5), (2,6), (3,7), (3,8), (4,5) and (4,6). Using these minimal cut sets, the reliability is calculated and found exactly similar to those in Figures 3 and 4.

In order to determine the maximum allowable replacement time, a study has been carried out for a given reliability of 0.5 at one month period for various failure rates. These results are shown in Figure 5. This figure again shows that the allowable failure rate has a significant effect on the allowable replacement time.

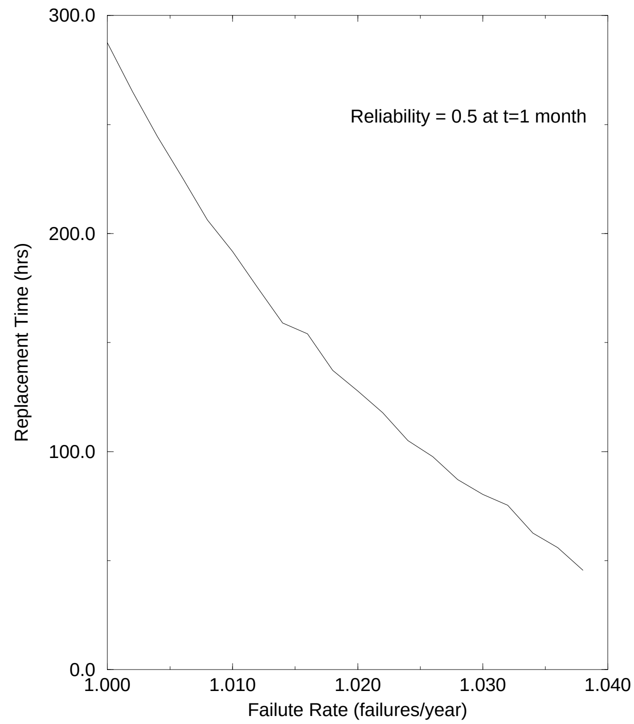

Figure 5: Replacement time for given reliability at one month.

5 Approximate Solution

The unreliability of a system can always be calculated precisely from the probability theory using the minimal cutsets. But the number of states that has to be evaluated increases tremendously as the number of minimal cut sets increases. It can be an exhaustive and time consuming exercise which can become prohibitive with large systems. Approximations can be made to overcome this situation. The results obtained with these approximations, although not very precise, permit rapid calculation of unreliability of a system. The degree of imprecision introduced is usually negligible and often within the tolerance associated with the data of the component reliabilities for system which have high values of component reliability.

Reliability of the SEN+ network with replacement have been analyzed using a set of appropriate but approximate equations developed based on the Markov model. Such approximate solution techniques use the concept of series and parallel configuration to first derive a set of equations, and these equations are then combined together in conjunction with the cut-set method [6].

Approximate equations for a system with n-components in series can be written as:

$$\lambda_{es} = \sum_{i=1}^{n} \lambda_i \tag{4}$$

$$r_{es} = \frac{\sum_{i=1}^{n} \lambda_i r_i}{\lambda_{es}} = \frac{U_{es}}{\lambda_{es}} \tag{5}$$

$$U_{es} = \lambda_{es} r_{es} = \sum_{i=1}^{n} \lambda_i r_i \tag{6}$$

where λ_{es}, r_{es} and U_{es} are the failure rate (f/year), repair time (hours/repair) and unavailability (hrs/year) of the *equivalent series* component. It should be noted that the equation for r_{es} is approximate for general cases but it is in fact an exact equation for those cases of series systems where the failure of other components can not occur when one component has failed. This situation occurs in practice when, after failure of the first component, the failure rates of the remaining operative but not working components either decrease to zero or become negligible. It should also be noted that the above equations are developed from the Markov model with underlying assumption of independent component failures. If the component behaviour is not independent then system state probabilities cannot normally be obtained by simple multiplication of the component state probabilities and the above equations can not be applied.

Approximate equation for parallel systems with n-components can be written as:

$$\mu_{ep} = \mu_1 + \mu_2 + \ldots + \mu_n \tag{7}$$

$$\frac{1}{r_{ep}} = \frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n} \tag{8}$$

But unlike the case of series systems, it is not possible to write a simple equation for equivalent λ_{ep} for a general n-component system.

The equation for two components in parallel can be used for certain systems to combine one component at a time. The equation for λ_{ep} for two-components in parallel is:

$$\lambda_{ep} = \frac{\lambda_1 \lambda_2 (r_1 + r_2)}{1 + \lambda_1 r_1 + \lambda_2 r_2} \tag{9}$$

Using the approximation $\lambda_i r_i << 1$ this can be reduced to

$$\lambda_{ep} \approx \lambda_1 \lambda_2 (r_1 + r_2) \tag{10}$$

Finally, for the case of n identical components in parallel, we have

$$\lambda_{ep} = \lambda^n \times n(r)^{n-1} = \lambda^n \times n(\text{MTTR of one component})^{n-1} \tag{11}$$

$$\mu_{ep} = n\mu \Rightarrow \frac{1}{r_{ep}} = \frac{n}{r} \Rightarrow r_{ep} = \frac{r}{n} \tag{12}$$

All the above equations assume that the individual component behaviours are statistically independent. The equivalent failure rate (λ_e) for the given SEN+ network with the previously mentioned 20 minimal cut sets can be calculated using the approximate Equations (4)-(12) as

$$\lambda_e = 8\lambda + 12\frac{2\lambda^2 r}{8760}. \tag{13}$$

In the above equation, it is assumed that all components have the same failure rate (λ f/year) and replacement time (r hours). The reliability equation for the system can be written as $R(t) = e^{-\lambda_e t}$. The approximate solution for the reliabilities are presented in Figure 6. It can be seen that the approximate method provides similar results when failure rate is small or replacement time is small. The difference between the exact and approximate solution for higher replacement time is not significant as can be seen from these figures.

6 Conclusion

The reliability of a multistage interconnection network with and without replacement has been studied in this paper. The results presented in this paper indicate that the

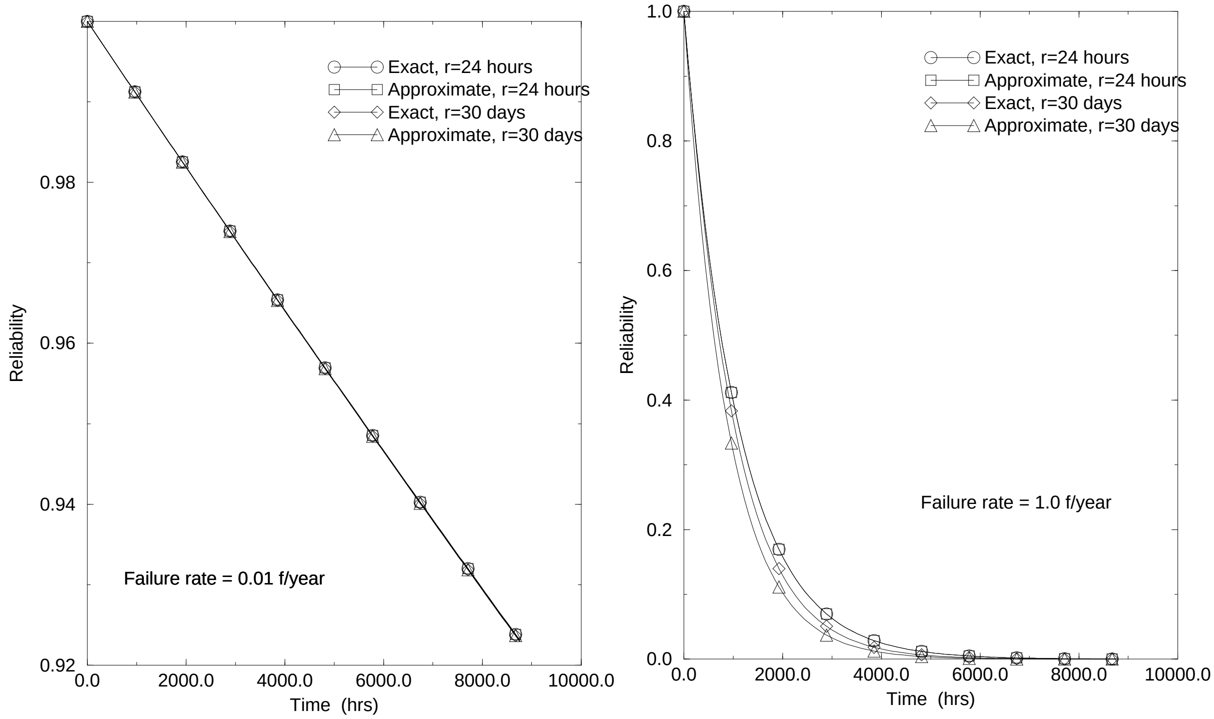

Figure 6: Approximate and exact solution for λ =0.01 and 1.0 failures/year.

replacement time has an effect on the overall system reliability. It should be noted here that the difference between the reliabilities with and without replacement is offset by the series nature of the switching elements in the first and the last stage of the network. The approximate solution using the cutset method has been demonstrated in this paper and found to provide satisfactory results when the failure rate and repair time are within realistic values.

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